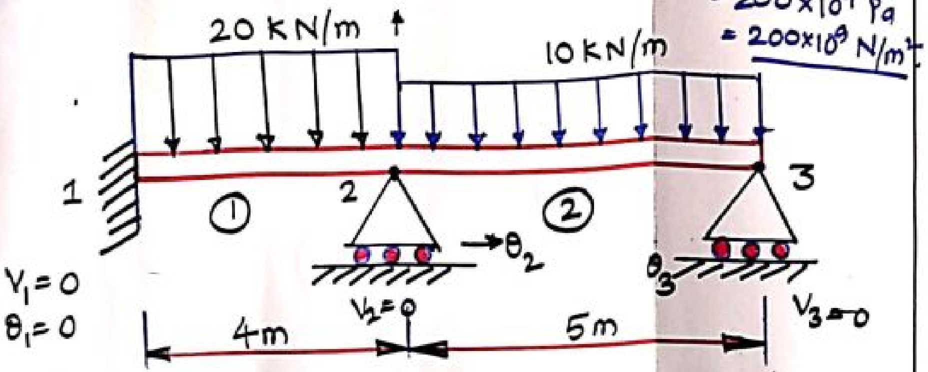
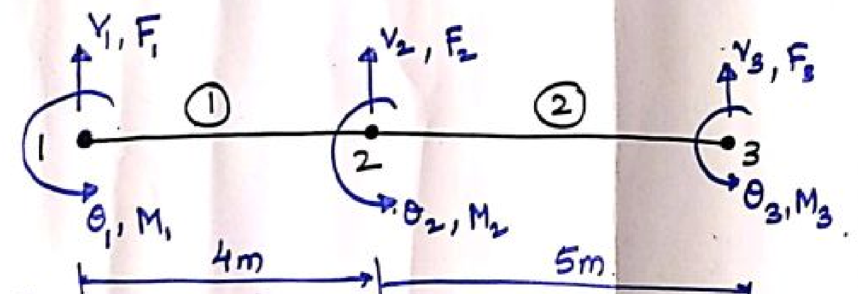
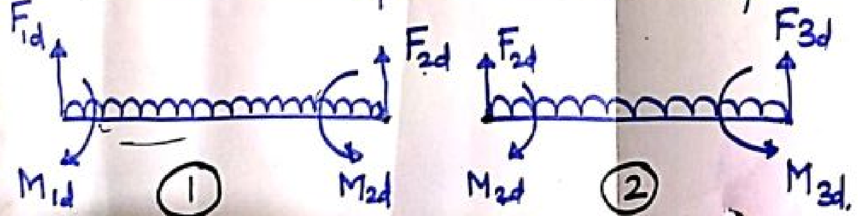
(4) FEA on Beams

For the beam shown in fig. Compute Slope at the Hinged Support point. E = 200 GPa, $I = 4\times10^{-6}\,m^4$. Use two beam Elements.

$E = 200\,GPa = 200\times10^9\,Pa = 200\times10^9\,N/m^2$







To Det. the Slope at Node (2) & (3). $[\theta_2, \theta_3]$:

The Element Stiffness matrix for (1).

$$[K_1] = \frac{E_1 I_1}{l_1^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \quad \text{——— (1)}$$

$$\frac{E_1 I_1}{l_1^3} = \frac{200\times10^9 \times 4\times10^{-6}}{(4)^3} = 12.5\times10^3\,N/m.$$

$$\frac{E_2 I_2}{(l_2)^3} = \frac{200\times10^9\times4\times10^{-6}}{(5)^3} = 6.4\times10^3\,N/m.$$

The Element Stiffness Equ^n for Element (1)

$$[K_1] = 12.5\times10^3 \begin{array}{c} \begin{array}{cccc} V_1 & \theta_1 & V_2 & \theta_2 \end{array} \\ \begin{bmatrix} 12 & 24 & -12 & 24 \\ 24 & 64 & -24 & 32 \\ -12 & -24 & 12 & -24 \\ 24 & 32 & -24 & 64 \end{bmatrix} \end{array} \begin{array}{c} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \end{array} = 10^3 \begin{array}{c} \begin{array}{cccc} V_1 & \theta_1 & V_2 & \theta_2 \end{array} \\ \begin{bmatrix} 150 & 300 & -150 & 300 \\ 300 & 800 & -300 & 400 \\ -150 & -300 & 150 & -300 \\ 300 & 400 & -300 & 800 \end{bmatrix} \end{array} \begin{array}{c} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \end{array} \quad \text{——— (3)}$$

$$[K_2] = 10^3 \begin{array}{c} \begin{array}{cccc} V_2 & \theta_2 & V_3 & \theta_3 \end{array} \\ \begin{bmatrix} 76.8 & 192 & -76.8 & 192 \\ 192 & 640 & -192 & 320 \\ -76.8 & -192 & 76.8 & -192 \\ 192 & 320 & -192 & 640 \end{bmatrix} \end{array} \begin{array}{c} V_2 \\ \theta_2 \\ V_3 \\ \theta_3 \end{array} \quad \text{——— (4)}$$

The Global Stiffness matrix $[K = K_1 + K_2]$.

$$= 10^3 \begin{array}{c} \begin{array}{cccccc} V_1 & \theta_1 & V_2 & \theta_2 & V_3 & \theta_3 \end{array} \\ \begin{bmatrix} 150 & 300 & -150 & 300 & 0 & 0 \\ 300 & 800 & -300 & 400 & 0 & 0 \\ -150 & -300 & 226.8 & -108 & -76.8 & 192 \\ 300 & 400 & -108 & 1440 & -192 & 320 \\ 0 & 0 & -76.8 & -192 & 76.8 & -192 \\ 0 & 0 & 192 & 320 & -192 & 640 \end{bmatrix} \end{array} \begin{array}{c} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \\ V_3 \\ \theta_3 \end{array} (6\times6)$$

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The Shear force at node ① & ②

$$F_{1d} = F_{2d} = \frac{w_1 l_1}{2} = \frac{20\times10^3 \times 4}{2} = 40000\,N$$

The B.M at node ① & ②

$$M_{1d} = M_{2d} = \frac{w_1 l_1^2}{12} = \frac{20\times10^3 \times (4)^2}{12} = 26667\,N\text{-}m$$

||ly For Element ②

The Force at ② & ③

$$F_{2d} = F_{3d} = \frac{w_2 l_2}{2} = \frac{10\times10^3 \times 5}{2} = 25000\,N$$

The B.M at Node ② & ③,

$$M_{2d} = M_{3d} = \frac{w_2 l_2^2}{12} = \frac{10\times10^3 \times (5)^2}{12} = 20833\,N\text{-}m$$

The resultant nodal Force & moment is

$$F = \begin{bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \\ F_3 \\ M_3 \end{bmatrix} = \begin{bmatrix} F_1 + F_{1d} \\ M_1 + M_{1d} \\ F_2 + F_{2d(1)} + F_{2d(2)} \\ M_2 + M_{2d(1)} + M_{2d(2)} \\ F_3 + F_{3d} \\ M_3 + M_{3d}. \end{bmatrix}$$

$F_{1d} = F_{2d(1)} = -40,000\,N$

$M_{1d} = -26\,667\,N\text{-}m$

$F_{2d(2)} = F_{3d} = -25000\,N$

$M_{2d(1)} = +26667\,N\text{-}m$

$M_{2d(2)} = -20,833\,Nm$

$M_{3d} = +20,833\,N\text{-}m$

Then the force Vector becomes

$$[F] = \begin{bmatrix} F_1 - 40000 \\ M_1 - 26667 \\ F_2 - 40000 - 25000 \\ M_2 + 26667 - 20833 \\ F_3 - 25000 \\ M_3 + 20833 \end{bmatrix} = \begin{bmatrix} F_1 - 40,000 \\ M_1 - 26667 \\ F_2 - 65000 \\ M_2 + 5834 \\ F_3 - 25000 \\ M_3 + 20833 \end{bmatrix}$$

The F.E Equation

$$[K][u] = [F]$$

The Boundary Conditions for the given fig is:

$V_1 = 0$, $\theta_1 = 0$, $V_2 = V_3 = 0$.

$F_2 = F_3 = 0$ & $M_2 = M_3 = 0$.

$$10^3 \begin{array}{c} \begin{array}{cccccc} V_1 & \theta_1 & V_2 & \theta_2 & V_3 & \theta_3 \end{array} \\ \begin{bmatrix} 150 & 300 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}_{(6\times6)} \end{array} \begin{bmatrix} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \\ V_3 \\ \theta_3 \end{bmatrix}_{(1\times6)} = \begin{bmatrix} F_1 - 40000 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}_{(1\times6)}$$

$$10^3 \begin{bmatrix} 1440 & 320 \\ 320 & 640 \end{bmatrix} \begin{bmatrix} \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 5834 \\ 20833 \end{bmatrix}$$

$$10^3 (1440\,\theta_2 + 320\,\theta_3) = 5834 \quad \text{---(i)}$$

$$10^3 (320\,\theta_2 + 640\,\theta_3) = 20833 \quad \text{(ii)}$$

$\theta_2 = -3.58 \times 10^{-3}$ rad

$\theta_3 = 34.3 \times 10^{-3}$ rad

Result :

The Slope at hinged Support

at Node ② $\theta_2 = -3.58\times10^{-3}$ rad

at node ③ $\theta_3 = 34.3\times10^{-3}$ rad.

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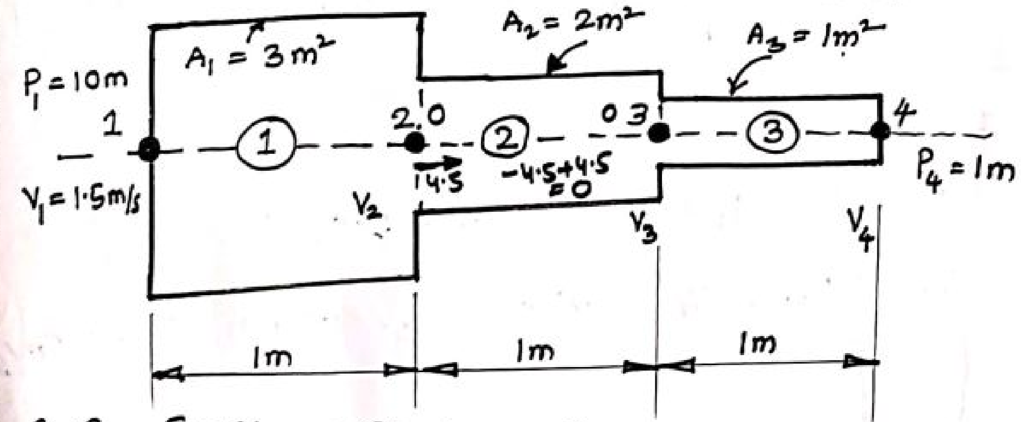
Fluid Flow Analysis :

For a Smooth pipe of Variable Cross-Section Shown in Fig. Determine the Potential at the Junction. The Potential at the left End is $P_1 = 10m$ & that at the right end is $P_4 = 1m$. If the Velocity of Fluid at the left end is 1.5 m/s, Find the Velocity at the Junctions & right End. Also Determine the Volumetric Flow rates.



Sol^n For Element ① $A_1 = 3m^2$, $l_1 = 1m$

② $A_2 = 2m^2$ $l_2 = 1m$

③ $A_3 = 1m^2$, $l_3 = 1m$.

For Smooth pipe, the permiabiliy $K = 1$

$K_1 = K_2 = K_3 = 1$ m/s.

Element Stiff. matrix (1).

$$[K_1] = \frac{K_1 A_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1\times 3}{1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{array}{cc} 1 & 2 \\ \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

IIIy Element ②

$$[K_2] = \frac{K_2 A_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1\times 2}{1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{array}{cc} 2 & 3 \\ \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array}$$

For Element ③

$$[K_3] = \frac{K_3 A_3}{l_3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1\times 1}{1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{array}{cc} 3 & 4 \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{array}{c} 3 \\ 4 \end{array} \end{array}$$

The Global Stiff. matrix for whole pipe,

$$[K] = [K_1] + [K_2] + [K_3]$$

$$[K] = \begin{array}{cc} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 5 & -2 & 0 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array} \quad \text{—— (I)}.$$

The Finite Element Equ^n for the Fluid flow problem

$$[K][P] = \{Q\} \text{ —(A)}$$

$P \rightarrow$ Nodal Potential
$Q \rightarrow$ Nodal flow rates

It is given that,

$P_1 = 10m$, $P_4 = 1m$ & $V_1 = 1.5 m/s$...

Volumetric flow rate, ($Q = AV$).

At Node 1, $Q_1 = A_1V_1 = 3m^2 \times 1.5m/s = 4.5 m^3/s$.

$$\boxed{Q_1 = Q_2 = Q_3 = Q_4 = 4.5 m^3/s}$$, Conservation of mass,

now, the Equ^n (A) Implies.

$$\begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 5 & -2 & 0 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 10 \\ P_2 \\ P_3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.5 \\ 0 \\ 0 \\ -4.5 \end{bmatrix}$$

Note:

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$$\begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 5 & -2 & 0 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ P_2 \\ P_3 \\ 1 \end{bmatrix} = \begin{Bmatrix} 4.5 \\ 0 \\ 0 \\ -4.5 \end{Bmatrix}$$

$$-30 + 5P_2 - 2P_3 = 0 \quad \text{— (1)}$$
$$-2P_2 + 3P_3 - 1 = 0 \quad \text{— (2)}$$

from (1). $5P_2 - 2P_3 = 30 \quad \times 2$

(2) $-2P_2 + 3P_3 = 1 \quad \times 5$

$$10\cancel{P_2} - 4P_3 = 60$$
$$-10\cancel{P_2} + 15P_3 = 5.$$
$$11P_3 = 65$$
$$P_3 = \frac{65}{11} = 5.91m$$

$\boxed{P_3 = 5.9m} \rightarrow$

$$10P_2 - 4P_3 = 60$$
$$10P_2 - 4\times(5.9) = 60$$

$\boxed{P_1 = 10m}$ $\boxed{P_2 = 8.36m}$, $\boxed{P_4 = 1m}$

V_1 1.5 m/s

Vel. at node 2 $\boxed{Q = AV.}$

$$V_2 = \frac{Q_2}{A_2} = \frac{4.5}{2} = 2.25 \text{ m/s}$$

$$V_3 = \frac{Q_3}{A_3} = \frac{4.5}{1} = 4.5 \text{ m/s}$$

$$V_4 = \frac{Q_4}{A_4} = \frac{4.5}{1} = 4.5 \text{ m/s}$$

Results:

$P_1 = 10$, $P_2 = 8.36$, $P_3 = 5.9$, $P_4 = 1$ m

$V_1 = 1.5$ m/s, $V_2 = 2.25$ m/s, $V_3 = 4.5$ m/s, $U_4 = 4.5$ m/s

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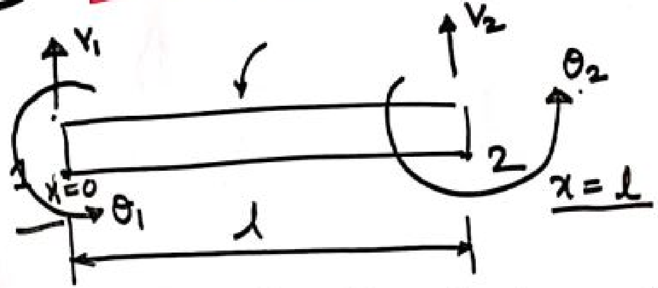
Shape Functions For Beams :

OR Hermite Shape Functions :



For the beam; the four displacement are two Vertical, two rotation (v_1, v_2 & θ_1, θ_2).

In the beam, 4 displacements, the Polynomial Functions used to represent the general displacement for the beam is given by

$$v(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3 \quad \text{—— (1).}$$

Where v is the tranversverse displacement & a_1, a_2, a_3, a_4 are Polynomial ceffients.

Equ^n (1) $\Rightarrow$ $$\frac{dv}{dx} = a_2 + 2x \cdot a_3 + 3x^2 a_4 \quad \text{—— (2).}$$

Applying the nodal Conditions Such that.

$v = v_1$, $\frac{dv}{dx} = \theta_1$, $x = 0$.

& $v = v_2$, $\frac{dv}{dx} = \theta_2$, $x = l$.

in Equ^n (1) & (2), we get:

$v_1 = a_1$ —— (3). $\qquad \theta_1 = a_2$ —— (4).

$v_2 = a_1 + a_2 l + a_3 l^2 + a_4 l^3$ —— (5)

$\theta_2 = a_2 + 2l a_3 + 3l^2 a_4$ —— (6).

Equ^n(5) $\Rightarrow$ $v_2 = v_1 + \theta_1 l + a_3 l^2 + a_4 l^3$

$a_3 l^2 + a_4 l^3 = v_2 - v_1 - \theta_1 l$ —— (7).

Equ^n(6) $\Rightarrow$ $2a_3 l + 3a_4 l^2 = \theta_2 - \theta_1$ —— (8).

(7)×3 $\quad 3a_3 l^2 + \cancel{3a_4 l^3} = 3v_2 - 3v_1 - 3\theta_1 l$. —— (9)

(8)×l $\quad -2a_3 l^2 \pm \cancel{3a_4 l^3} = -\theta_2 l \mp \theta_1 l$ —— (10).

Equ^n (9)−(10) $\quad + a_3 l^2 = 3v_2 - 3v_1 - 3\theta_1 l - \theta_2 l + \theta_1 l$

$a_3 l^2 = 3v_2 - 3v_1 - 2\theta_1 l - \theta_2 l$

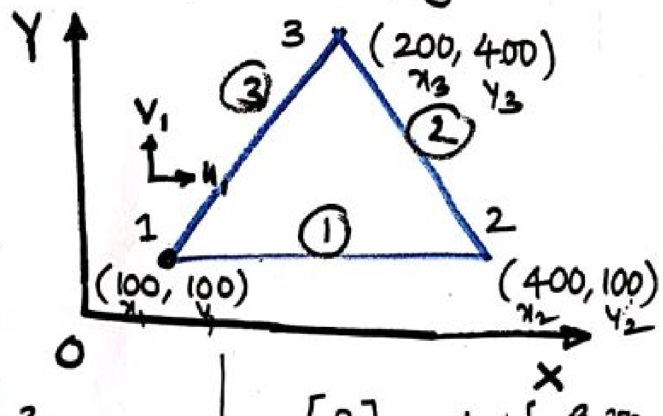
$$\boxed{a_3 = \frac{3}{l^2}(v_2 - v_1) - \frac{1}{l}(2\theta_1 + \theta_2)}$$

Equ^n(7), $\quad a_4 l^3 = v_2 - v_1 - \theta_1 l - a_3 l^2$

2D (Vector Variable) Problems :

For the Plane Stress element shown in fig, the nodal displacements are

1) $u_1 = 2\,mm$, $V_1 = 1\,mm$
2) $u_2 = 1\,mm$, $V_2 = 1.5\,mm$
3) $u_3 = 2.5\,mm$, $V_3 = 0.5\,mm$



$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$\boxed{A = 45{,}000\ mm^2}$$

Determine the Element Stresses (

Assuming, $E = 200\,GN/m^2$, $\mu = 0.3$

$t = 10\,mm$. All Co-ordinates in mm.

σ_1, σ_2 & σ_3 = ? $\quad E = 200\times10^3\ N/mm^2$

$$\boxed{E = 2\times10^5\ N/mm^2}$$

Soln

$[\sigma] = [D]\cdot[e] \rightarrow$ Strain-Displ.

$[\sigma] = [D][B][\delta]$

for the Plane ~~Strain~~ Stress $\longrightarrow$

$$[D] = \frac{E}{(1-\mu^2)}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \left(\frac{1-\mu}{2}\right) \end{bmatrix}$$

$$[D] = \frac{2\times10^6}{91}\begin{bmatrix} 10 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & 3.5 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \begin{bmatrix} -879.12 \\ -708.16 \\ 341.86 \end{bmatrix}$$

$\sigma_1 = -879\ N/mm^2$

$\sigma_2 = -708\ N/mm^2$

$\sigma_3 = 341.88\ N/mm^2$

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 = -300 & 0 & \beta_2 & 0 & \beta_2 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & V_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \quad \text{—— (2)}$$

$\beta_1 = y_2 - y_3$, $\beta_2 = y_3 - y_1$, $\beta_3 = y_1 - y_2$, $\gamma_1 = x_3 - x_2$, $\gamma_2 = x_1 - x_3$, $\gamma_3 = x_2 - x_1$

$$= \frac{1}{2A}\begin{bmatrix} -300 & 0 & 300 & 0 & 0 & 0 \\ 0 & -200 & 0 & -100 & 0 & 300 \\ -200 & -300 & -100 & 300 & 300 & 0 \end{bmatrix} = \frac{1}{900}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 3 \\ -2 & -3 & -1 & 3 & 3 & 0 \end{bmatrix}$$

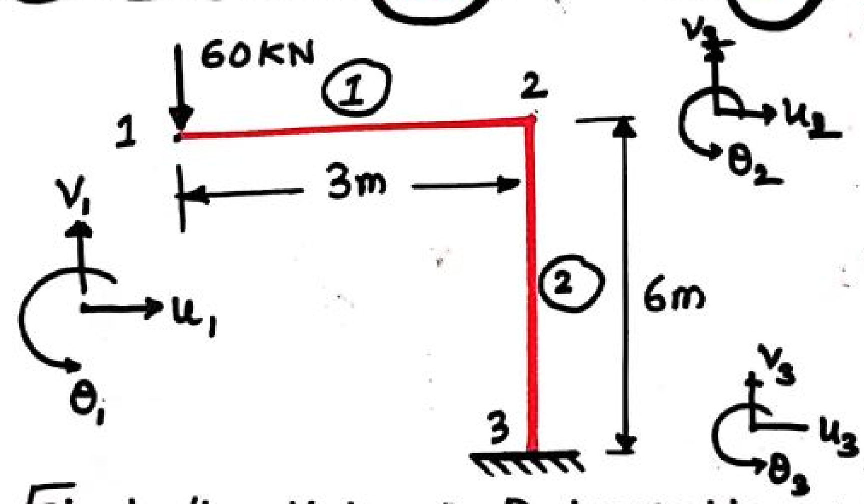
$$\delta = \begin{bmatrix} u_1 \\ V_1 \\ u_2 \\ V_2 \\ u_3 \\ V_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1.5 \\ 2.5 \\ 0.5 \end{bmatrix}$$

$[\sigma] = [D][B]\cdot[\delta]$

$$= \frac{2\times10^6}{91}\Big[\ ①\ \Big]_{(3\times3)} \times \frac{1}{900}\Big[\ ②\ \Big]_{(6\times3)} \times \Big[\quad\Big]_{(1\times6)}$$

$$= \frac{2\times10^6}{91\times900}\begin{bmatrix} 10 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & 3.5 \end{bmatrix}_{(3\times3)} \times \begin{bmatrix} -3 \\ -2 \\ 4 \end{bmatrix}$$

Analysis of Frames :



Find the Unknown Deformation at nodes 1 and 2 For the Frame.

Use $E = 200\,GPa = 200\times10^9\ N/m^2 = 2\times10^{11}\ N/m^2$

$A = 0.0112\ m^2$

$I = 2.39\times10^{-5}\ m^2$.

The Stiffness matrix for each element 1

1 $\theta = 0^\circ$ $\sin\theta = \sin 0 = 0$

$\cos\theta = \cos 0 = 1$

$C = 1,\quad S = 0$

$[k] = \cancel{\left(AC^2+\frac{12I}{l^2}S^2\right)\ \left(A-\frac{12I}{l^2}\right)cs\ \ -\frac{6I}{l}s\ \ -\left(AC^2+\frac{12I}{l^2}S^2\right)\ -\left(A-\frac{12I}{l^2}\right)cs\ \ -\frac{6I}{l}s}$

A−

$$[k] = \frac{E}{l}\begin{bmatrix} \left(AC^2+\frac{12I}{l^2}S^2\right) & \left(A-\frac{12I}{l^2}\right)CS & -\frac{6I}{l}S & -\left(AC^2+\frac{12I}{l^2}S^2\right) & -\left(A-\frac{12I}{l^2}\right)CS & -\frac{6I}{l}S \\ \left(A-\frac{12I}{l^2}\right)CS & \left(AS^2+\frac{12I}{l^2}C^2\right) & \frac{6I}{l}C & -\left(A-\frac{12I}{l^2}\right)CS & -\left(AS^2+\frac{12I}{l^2}C^2\right) & \frac{6I}{l}C \\ -\frac{6I}{l}S & \frac{6I}{l}C & 4I & \frac{6I}{l}S & -\frac{6I}{l}C & 2I \\ -\left(AC^2+\frac{12I}{l^2}S^2\right) & -\left(A-\frac{12I}{l^2}\right)CS & \frac{6I}{l}S & \left(AC^2+\frac{12I}{l^2}S^2\right) & \left(A-\frac{12I}{l^2}\right)CS & \frac{6I}{l}S \\ -\left(A-\frac{12I}{l^2}\right)CS & \left(AS^2+\frac{12I}{l^2}C^2\right) & -\frac{6I}{l}C & \left(A-\frac{12I}{l^2}\right)CS & \left(AS^2+\frac{12I}{l^2}C^2\right) & -\frac{6I}{l}C \\ -\frac{6I}{l}S & \frac{6I}{l}C & 2I & \frac{6I}{l}S & -\frac{6I}{l}C & 4I \end{bmatrix}_{6\times6}$$

$$\frac{E_1}{l_1} = \frac{2\times10^{11}}{3} = 6.67\times10^{10}.$$

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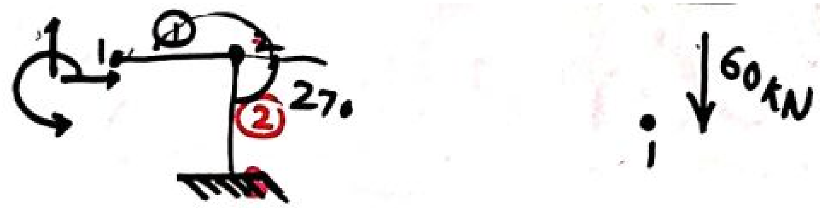
$$
\begin{array}{c|cccccc|c}
 & u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 & \\
\hline
 & 0.0112 & 0 & 0 & -0.0112 & 0 & 0 & u_1 \\
 & 0 & 3.19\times10^{-5} & 4.7\times10^{-5} & 0 & -3.19\times10^{-5} & 4.7\times10^{-5} & v_1 \\
 & 0 & 4.7\times10^{-5} & 9.5\times10^{-5} & 0 & -4.78\times10^{-5} & 4.7\times10^{-5} & \theta_1 \\
=6.67\times10^{10} & -0.0112 & 0 & 0 & 0.0112 & 0 & 0 & u_2 \\
 & 0 & -3.19\times10^{-5} & 4.7\times10^{-5} & 0 & 3.19\times10^{-5} & 4.78\times10^{-5} & v_2 \\
 & 0 & -4.7\times10^{-5} & 4.78\times10^{-5} & 0 & 4.78\times10^{-5} & 9.56\times10^{-5} & \theta_2
\end{array}
\quad (6\times6)
$$



▷ Formation for FEM

$$[K][u] = [F]$$

$$[K]=[K_1]+[K_2]$$

$u_1\ v_1\ \theta_1\ u_2\ v_2\ \theta_2\ u_3\ v_3\ \theta_3$

$$\begin{bmatrix} \ \end{bmatrix}_{(9\times9)}$$

$$
\begin{bmatrix} \ \\ \ \\ \ \end{bmatrix}_{(9\times9)}
\begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \end{bmatrix}_{(1\times9)}
=
\begin{bmatrix} F_{1x} \\ F_{1y} \\ M_1 \\ F_{2x} \\ F_{2y} \\ M_2 \\ F_{3x} \\ F_{3y} \\ M_3 \end{bmatrix}_{(1\times9)}
$$

$F_{1y} \leftarrow -60\times10^3$

Stiffness Matrix for Element ②

$C = \cos 270 = 0 \qquad S = \sin 270 = -1$

$\theta = 270°,$

$$\boxed{C = 0, \quad S = -1}$$

$$
[K_2] = \frac{E}{l}
\begin{array}{|cccccc|c}
u_2 & v_2 & \theta_2 & u_3 & v_2 & \theta_3 & \\
\hline
7.79\times10^{-6} & 0 & 2.39\times10^{-5} & -7.7\times10^{-6} & 0 & 2.3\times10^{-5} & u_2 \\
0 & 0.0112 & 0 & 0 & -0.112 & 0 & v_2 \\
2.39\times10^{-5} & 0 & 9.5\times10^{-5} & -2.39\times10^{-5} & 0 & 4.78\times10^{-5} & \theta_2 \\
-7.9\times10^{-1} & 0 & 2.39\times10^{-5} & 7.7\times10^{-6} & 0 & 2.3\times10^{-5} & u_3 \\
0 & -0.0112 & 0 & 0 & 0.0012 & 0 & v_3 \\
2.39\times10^{-5} & 0 & 4.78\times10^{-5} & 2.39\times10^{-5} & 0 & 9.56\times10^{-5} & \theta_3
\end{array}
\quad (6\times6)
$$

What are the B.C,

$u_3 = v_3 = \theta_3 = 0, \quad F_{1y} = -60{,}000\ N$

$$\boxed{u_1, v_1, \theta_1, \ u_2, v_2, \theta_2 = ?}$$

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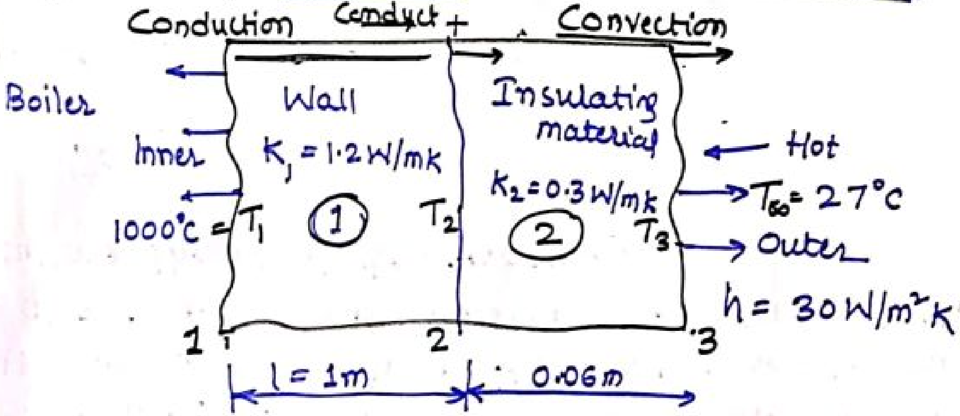
Heat transfer Problems:

A Wall of 1m thickness has thermal Conductivity of 1·2 W/mK. The wall is to be insulted with a material of thickness 0·06m having an average thermal conductivity of 0·3 W/mK. The inner Surface temperature is 1000°c. and Outside of the insultion is exposed to atmospheric air at 27°c with heat transfer-co-efficient of 30 W/m²K. Calcutate nodal temperature.



Let now, $K_1 = 1{\cdot}2\,W/mK$, $K_2 = 0{\cdot}3\,W/mK$

$l_1 = 1m$ $\quad l_2 = 0{\cdot}06\,m$

T_1, T_2, T_3 are the temp at 1, 2, 3 resp.

$T_1 = 1000°c = 1000 + 273 = 1273\,K.$

$T_\infty = 27°c = 27 + 273 = 300\,K.$

$h = 30\,W/m^2K$

Sol^n $\quad T_2$ & T_3 at node 2 & node 3;

The F.E. Equation is

$$[K][T] = [F]$$

$A = 1m^2$

$$K_1 = \frac{K_1A_1}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{1{\cdot}2\times 1}{1}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \end{array}\begin{array}{c}1\\2\end{array} = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix}1{\cdot}2 & -1{\cdot}2\\ -1{\cdot}2 & 1{\cdot}2\end{bmatrix} \end{array}\begin{array}{c}1\\2\end{array}$$

$||^{ly}$ Conduction 2

$$[K_2]_2 = \frac{K_2A}{l_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{0{\cdot}3\times 1}{0{\cdot}06}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{0{\cdot}3}{0{\cdot}06}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

Convection at node 3

$$[K_2]_3 = hA\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} = 30\times 1\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 30\end{bmatrix}$$

The Combined thermal Stiffness matrix Element 2

$$K_2 = [K_2]_2 + [K_2]_3 = \frac{0{\cdot}3}{0{\cdot}06}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 30\end{bmatrix}$$

$$K_2 = \begin{bmatrix}5 & -5\\ -5 & 5\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 30\end{bmatrix} = \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix}5 & -5\\ -5 & 35\end{bmatrix} \end{array}$$

The Global Stiffness matrix for $K = K_1 + K_2$

$$[K] = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix}1{\cdot}2 & -1{\cdot}2 & 0\\ -1{\cdot}2 & 6{\cdot}2 & -5\\ 0 & -5 & 35\end{bmatrix} \end{array}\begin{array}{c}1\\2\\3\end{array}\begin{bmatrix}T_1\\ T_2\\ T_3\end{bmatrix} = \begin{bmatrix}\cancel{QA}\\ 0\\ hAT_\infty\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 9000\end{bmatrix}$$

1·2
$\frac{5{\cdot}2}{6{\cdot}2}$ = 30×1×300
$hAT_\infty = 9000\,W$

$$\boxed{\begin{aligned} -1{\cdot}2\,T_1 + 6{\cdot}2\,T_2 - 5T_3 &= 0\\ 0 - 5\,T_2 + 35\,T_3 &= 9000\end{aligned}}$$

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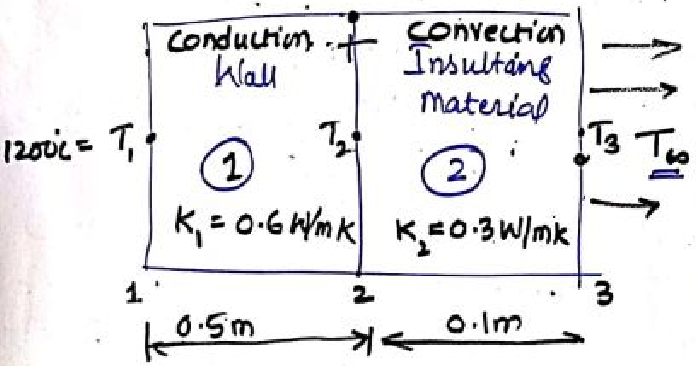
Heat transfer

A Wall of 0.5m thickness having thermal Conductivity of 0.6 W/mK. The wall is to be insultated with a material of thickness 0.1m having an average thermal Conductivity of 0.3 W/mK. The Inner Surface temperature is 1200°c & the Outside of the insultation is Exposed to atm air at 30°c with heat transfer Co-efficient of 40W/m²K. Calcultate the nodal temperature.



Let assume
$A_1 = A_2 = 1m^2$

$h = 40\ W/m^2K$.

$T_1 = 1200°c = 1200 + 273 = 1473\ K$.

$T_2 = ?$ & $T_3 = ?$

For the Element (1) . $K_1 = \cancel{0.6 W/mK} = 6\ W/mK$.

$l_1 = 0.5m$

For Element (2) $K_2 = 0.3\ W/mK$, $l_2 = 0.1m$.

$h = 40\ W/m^2K$.

Sol^n Let consider (1)

The thermal Stiffness matrix is

$$[K_1] = \frac{K_1 A_1}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{6\times 1}{0.5}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{array}{c|c} \begin{matrix}1 & \quad 2\end{matrix} & \\ \begin{bmatrix}12 & -12\\ -12 & 12\end{bmatrix} & \begin{matrix}1\\2\end{matrix}\end{array}$$

for Element (2) the conduction part of the s.m.

$$[K_2]_1 = \frac{K_2 A_2}{l_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{0.3\times 1}{0.1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}3 & -3\\ -3 & 3\end{bmatrix}$$

The Convection part of. ther. Sti. matrix

$$[K_2]_2 = hA\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} = 40\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 40\end{bmatrix}$$

The Combined thermal Stiff. matrix (2).

$$[K_2] = [K_2]_1 + [K_2]_2 = \begin{bmatrix}3 & -3\\ -3 & 3\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 40\end{bmatrix}$$

$$[K_2] = \begin{array}{cc} \begin{matrix}2 & \quad 3\end{matrix} & \\ \begin{bmatrix}3 & -3\\ -3 & 43\end{bmatrix} & \begin{matrix}2\\3\end{matrix}\end{array}, \quad [K_1] = \begin{array}{cc} \begin{matrix}1 & \quad 2\end{matrix} & \\ \begin{bmatrix}12 & -12\\ -12 & 12\end{bmatrix} & \begin{matrix}1\\2\end{matrix}\end{array}$$

The global Stiffness matrix

$$[K] = \begin{array}{cc} \begin{matrix}1 & \quad 2 & \quad 3\end{matrix} & \\ \begin{bmatrix}12 & -12 & 0\\ -12 & 15 & -3\\ 0 & -3 & 43\end{bmatrix} & \begin{matrix}1\\2\\3\end{matrix}\end{array} \qquad \text{—— (1).}$$

(3×3)

The F.E. equation is, $[K][T] = [F]$

$$\begin{bmatrix}12 & -12 & 0\\ -12 & 15 & -3\\ 0 & -3 & 43\end{bmatrix}\begin{bmatrix}T_1\\ T_2\\ T_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 12120\end{bmatrix}$$

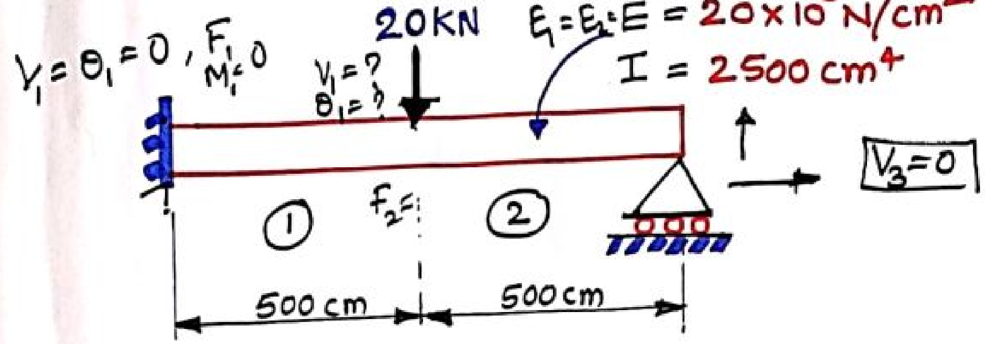
$30 + 273 = 303$

$hAT_\infty = 40\times 1\times 303$

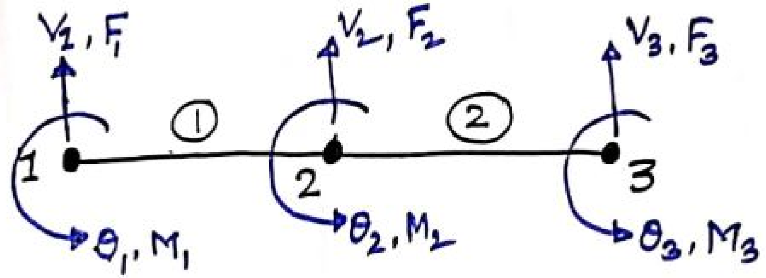
$\boxed{F = 12120\,W}$

Analysis of Beams

A Beam, Fixed at One End and Supported By a roller at the Other end, has a 20 kN Concentrated load applied at the Centre of the Span, as Shown in fig. Calculate the deflection Under the load.



Let the beam is divided into two parts,



<u>Step 1</u>: <u>nodal deflection</u>

For Element 1

$$[K] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \quad \text{— (I)}$$

$$[K_1] = \frac{E_1 I_1}{l_1^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

now, $\frac{E_1 I_1}{l_1^3} = \frac{20 \times 10^6 \times 2500}{(500)^3}$

$= 400\ N/cm$

$$\therefore [K_1] = 400 \begin{array}{c} \begin{array}{cccc} V_1 & \theta_1 & V_2 & \theta_2 \end{array} \\ \begin{bmatrix} 12 & 3000 & -12 & 3000 \\ 3000 & 10^6 & -3000 & 5\times10^5 \\ -12 & -3000 & 12 & -3000 \\ 3000 & 5\times10^5 & -3000 & 10^6 \end{bmatrix} \end{array} \begin{array}{c} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \end{array} \quad \text{— (1)}.$$

Ily for Element 2.

$$[K_2] = \frac{E_2 I_2}{(l_2)^3}\begin{bmatrix} & & & \\ & " & & \\ & & & \end{bmatrix}, \quad l = 500\ cm.$$

$$[K_2] = 400 \begin{array}{c} \begin{array}{cccc} V_2 & \theta_2 & V_3 & \theta_3 \end{array} \\ \begin{bmatrix} 12 & 3000 & -12 & 3000 \\ 3000 & 10^6 & -3000 & 5\times10^5 \\ -12 & -3000 & 12 & -3000 \\ 3000 & 5\times10^5 & -3000 & 10^6 \end{bmatrix} \end{array} \begin{array}{c} V_2 \\ \theta_2 \\ V_3 \\ \theta_3 \end{array} \quad \text{— (2)}$$

<u>Step 2</u>:

<u>Global stiffness matrix</u>:

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The global stiffness matrix is

$$
[K] = 400 \begin{array}{c}
\begin{array}{cccccc} V_1 & \theta_1 & V_2 & \theta_2 & V_3 & \theta_3 \end{array} \\
\begin{bmatrix}
12 & 3000 & -12 & 3000 & 0 & 0 \\
3000 & 10^6 & -3000 & 5\times10^5 & 0 & 0 \\
-12 & -3000 & 24 & 0 & -12 & 3000 \\
3000 & 5\times10^5 & 0 & 2\times10^6 & -3000 & 5\times10^5 \\
0 & 0 & -12 & -3000 & 12 & -3000 \\
0 & 0 & 3000 & 5\times10^5 & -3000 & 10^6
\end{bmatrix}
\end{array}
\begin{array}{c} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \\ V_3 \\ \theta_3 \end{array}
\begin{bmatrix} 0 \\ 0 \\ V_2 \\ \theta_2 \\ 0 \\ \theta_3 \end{bmatrix}
=
\begin{bmatrix} F_1 \\ M_1 \\ -20{,}000 \\ 0 \\ F_3 \\ 0 \end{bmatrix}
$$

[K] [u] [F]

(6x6)

The Formation for FE Equation

$[K][u] = [F]$

Boundary Conditions & loading Conditions,

$V_1 = 0,\ \theta_1 = 0,\ V_3 = 0,\ F_2 = -20\times10^3\,N,\ M_2 = 0,\ M_3 = 0.$

$$
400 \begin{bmatrix} 24 & 0 & +3000 \\ 0 & 2\times10^6 & 5\times10^5 \\ 3000 & 5\times10^5 & 10^6 \end{bmatrix} \begin{bmatrix} V_2 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} -20000 \\ 0 \\ 0 \end{bmatrix}
$$

$400(24V_2 + 3000\theta_3) = -20{,}000.$ — (i)

$400(2\times10^6\theta_2 + 5\times10^5\theta_3) = 0.$ — (ii)

$400(3000V_2 + 5\times10^5\theta_2 + 10^6\theta_3) = 0$ — (iii)

$\theta_3 = 0.0125$ rad

$\theta_2 = -0.003125$ rad

$V_2 = -3.646$ cm

Node 1

$F_1 = 400(-12V_2 + 3000\theta_2)$

$= 400(-12\times-3.646 + 3000\times0.003125)$

$\boxed{F_1 = 13750N}$

Bending moment at node 1

$M_1 = 400[-3000V_2 + 5\times10^5\theta_2]$

$= 400[-3000(-3.646) + 5\times10^5\times(-0.003125)]$

$\boxed{M_1 = 3750200\ N\text{-}cm}$

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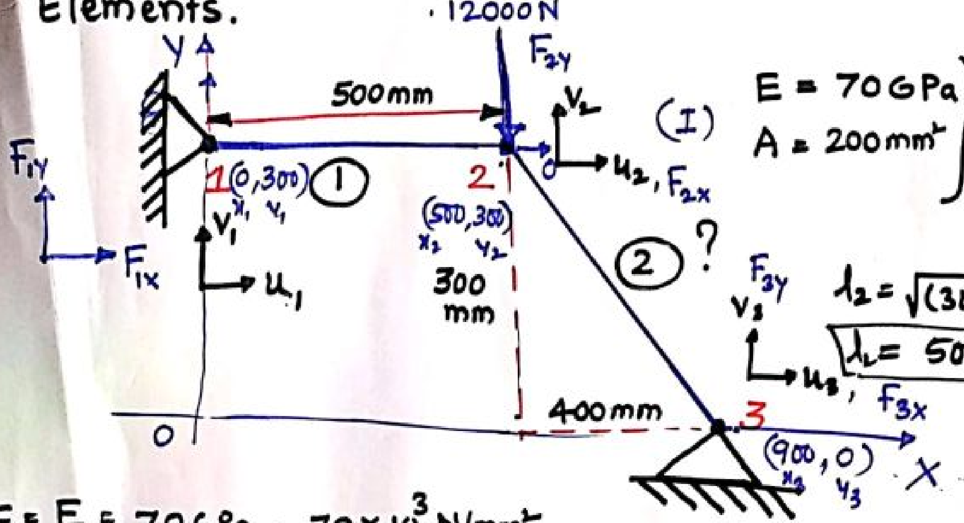
Analysis of Trusses

For the two-bar truss as shown in fig. Determine the displacements at node 2 & the Stresses in both Elements.



(I) $E = 70\,GPa$, $A = 200\,mm^2$ } For Both Members

$l_2 = \sqrt{(300)^2 + (400)^2}$

$\boxed{l_2 = 500\ mm}$

$E_1 = E_2 = 70\,GPa = 70 \times 10^3\ N/mm^2$

$A_1 = A_2 = 200\ mm^2$.

For Element 1:

$$[k] = \frac{AE}{l}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & s^2 & cs & s^2 \end{bmatrix}$$

$l = c = \cos\theta$

$s = m = \sin\theta$

$\theta = 0$

$\boxed{\cos\theta = 1}$

$\sin\theta = 0$

Geometry,

$$= \frac{AE_1}{l_1} = \frac{200 \times 70 \times 10^3}{500} = 28 \times 10^3\ N/mm,\quad \cos\theta_1 = \frac{x_2 - x_1}{l_1} = \frac{500}{500}$$

$\boxed{\cos\theta_1 = 1}$

$\boxed{\sin\theta = 0}$

$$[k_1] = 28 \times 10^3 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array} \quad \text{— (1)}$$

||^{ly} for Element ②

$$[k_2] = \frac{A_2 E_2}{l_2} \begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix} \end{array} \begin{array}{c} u_2 \\ v_2 \\ u_3 \\ v_3 \end{array} \quad \text{— (2)}$$

$$c = \cos\theta_2 = \frac{x_3 - x_2}{l_2} = \frac{400}{500} = 0.8$$

$$s = \sin\theta_2 = \frac{y_3 - y_2}{l_2} = \frac{-300}{500} = -0.6$$

$$= \frac{200 \times 70 \times 10^3}{500} = 28 \times 10^3\ N/mm$$

> The Global Stiffness Matrices (Assembly)

$$(K) = [k_1] + [k_2]$$

$$[K] = 28 \times 10^3 \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1.64 & -0.48 & -0.64 & 0.48 \\ 0 & 0 & -0.48 & 0.36 & 0.48 & -0.36 \\ 0 & 0 & -0.64 & 0.48 & 0.64 & -0.48 \\ 0 & 0 & 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}$$

The FE Equation is, $[K][u] = [F]$

$$\{u\} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}, \quad [F] = \begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{bmatrix}$$

Boundary Conditions

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$$28\times 10^3 \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1.64 & -0.48 & -0.64 & 0.48 \\ 0 & 0 & -0.48 & 0.36 & 0.48 & -0.36 \\ 0 & 0 & -0.64 & 0.48 & 0.64 & -0.48 \\ 0 & 0 & 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix} \end{array} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} = 10^3 \begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ -12000 \\ F_{3x} \\ F_{3y} \end{bmatrix}$$

The Boundary Conditions are.

$u_1=0,\ v_1=0,\quad u_3=0,\ v_3=0,\quad F_{2x}=0,\ F_{2y}=-12000$

$$28\begin{bmatrix} 1.64 & -0.48 \\ -0.48 & 0.36 \end{bmatrix}\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -12 \end{bmatrix}$$

$$28(1.64\,u_2 - 0.48\,v_2) = 0.$$

$$28(-0.48\,u_2 - 0.36\,v_2) = -12.$$

$$\boxed{\begin{aligned} u_2 &= -0.57\ \text{mm} \\ v_2 &= -1.96\ \text{mm} \end{aligned}}$$

1 $\quad \sigma_1 = \dfrac{E_2}{l_1}\begin{bmatrix} -c & -s & c & s \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$

$$\boxed{\sigma_1 = -79.8\ \text{N/mm}^2}$$

$$\sigma_2 = \frac{E_2}{l}\begin{bmatrix} -c & -s & c & s \end{bmatrix}\begin{bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} = \boxed{-100.8\ \text{N/mm}^2}$$

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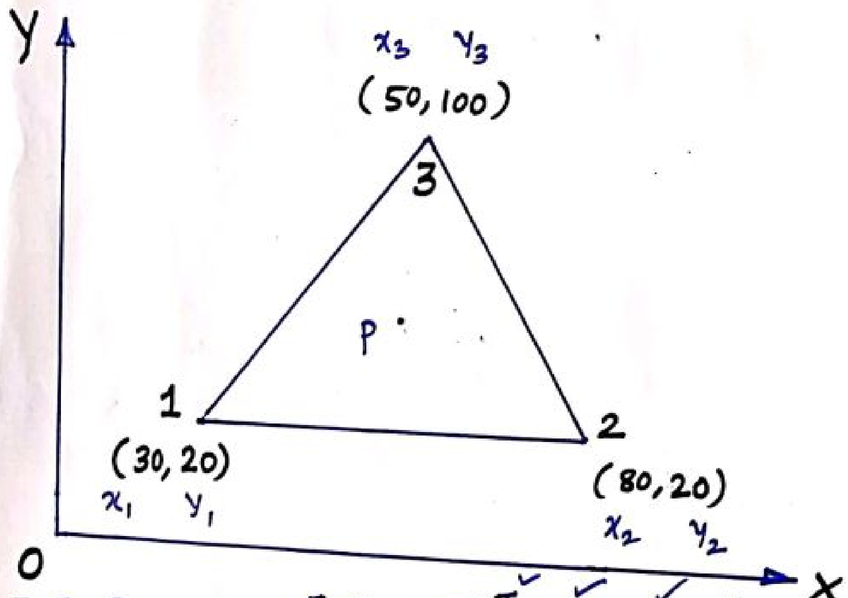
Two dimensional Element Stiffness Matrix for CST

P-2 For the Plane Stress Element shown in fig. Evalute the Stiffness matrix. Assume modulus of Elasticity $E = 210\times10^3\ N/mm^2$, Poisson's ratio $\mu = 0.25$ & Element thickness $t = 10\,mm$. The Co-ordinates are given in millimetres.



$[D] = ?$

$[K] = ?$

$E = 210\times10^3\ N/mm^2.$

$\mu = 0.25.$

$[K] = [B]^T \cdot [D] \cdot [B] \cdot A \cdot t$

$[B]$ - Strain - Disp. Matrix

$[D]$ - Stress-Strain matrix

Ref. the fig.

$x_1 = 30$ $\quad y_1 = 20$

$x_2 = 80$ $\quad y_2 = 20$

$x_3 = 50$ $\quad y_3 = 100$

Area of the triangle.

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 30 & 20 \\ 1 & 80 & 20 \\ 1 & 50 & 100 \end{vmatrix}$$

$$= \frac{1}{2}\left[1(8000-1000) - 30(100-20) + 20(50-80)\right)$$

$$A = \frac{1}{2}\left[7000 - 2400 + 600\right) = 2000\ mm^2.$$

The Strain-Displacement Matrix $[B]$.

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$\beta_1 = y_2 - y_3 = 20-100 = -80$, $\quad \gamma_1 = x_3 - x_2 = 50-80 = -30$

$\beta_2 = y_3 - y_1 = 100-20 = 80$, $\quad \gamma_2 = x_1 - x_3 = 30-50 = -20$

$\beta_3 = y_1 - y_2 = 20-20 = 0$, $\quad \gamma_3 = x_2 - x_1 = 80-30 = 50.$

$$[B] = \frac{1}{2\times2000}\begin{bmatrix} -80 & 0 & 80 & 0 & 0 & 0 \\ 0 & -30 & 0 & -20 & 0 & 50 \\ -30 & -80 & -20 & 80 & 50 & 0 \end{bmatrix}$$

$$[B] = \frac{1\cancel{0}}{400\cancel{0}}\begin{bmatrix} -8 & 0 & 8 & 0 & 0 & 0 \\ 0 & -3 & 0 & -2 & 0 & 5 \\ -3 & -8 & -2 & 8 & 5 & 0 \end{bmatrix}$$

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$$[B]^T = \frac{1}{400}\begin{bmatrix} -8 & 0 & -3 \\ 0 & -3 & -8 \\ 8 & 0 & -2 \\ 0 & -2 & 8 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{bmatrix}$$

The Stress- Strain relation matrix for <u>Plane Stress</u>

$$[D] = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \left(\frac{1-\mu}{2}\right) \end{bmatrix}$$ for Plane Stress.

$$= \frac{210\times 10^3}{1-(0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \left(\frac{1-0.25}{2}\right) \end{bmatrix}$$

$$= \frac{210\times 10^3}{0.9375}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \left(\frac{1-0.25}{2}\right) \end{bmatrix}$$

$$= \left(\frac{210\times 10^3\times 0.25}{0.9375}\right)\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[D] = 56\times 10^3\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \quad \text{—— (ii)}$$

for Plane Strain

$$[D] = \frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix} (1-\mu) & \mu & 0 \\ \mu & (1-\mu) & 0 \\ 0 & 0 & \left(\frac{1-2\mu}{2}\right) \end{bmatrix} \text{ – Plane Strain,}$$

$[K] = [B]^T [B] [D]\, A \cdot t$.

$$= \frac{1}{400}\begin{bmatrix} -8 & 0 & -3 \\ 0 & -3 & -8 \\ 8 & 0 & -2 \\ 0 & -2 & 8 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{bmatrix}_{(6\times 3)} \times \frac{1}{400}\begin{bmatrix} -8 & 0 & 8 & 0 & 0 & 0 \\ 0 & -3 & 0 & -2 & 0 & 5 \\ -3 & -8 & -2 & 8 & 5 & 0 \end{bmatrix}_{(3\times 6)} \times 56\times 10^3\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}_{(3\times 3)}$$

$$\times 2000 \times 10.$$

$$= \left(\frac{56\times 10^3 \times 2000\times 10}{400\times 400}\right)\begin{bmatrix} -8 & 0 & -3 \\ 0 & -3 & -8 \\ 8 & 0 & -2 \\ 0 & -2 & 8 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{bmatrix}_{(6\times 3)} \times \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}_{(3\times 3)} \times \begin{bmatrix} -8 & 0 & 8 & 0 & 0 & 0 \\ 0 & -3 & 0 & -2 & 0 & 5 \\ -3 & -8 & -2 & 8 & 5 & 0 \end{bmatrix}_{(3\times 6)}.$$

$(6\times 3) \times (3\times 6).$

$$[K] = 7\times 10^3\begin{bmatrix} 269.5 & 60 & -247 & -20 & -22.5 & -40 \\ 60 & 132 & 0 & -72 & -60 & -60 \\ 247 & 0 & 262 & -40 & -15 & 40 \\ -20 & -72 & -40 & 112 & 60 & -40 \\ -22.5 & -60 & -15 & 60 & 37.5 & 0 \\ -40 & -60 & 40 & -40 & 0 & 100 \end{bmatrix} \text{ N/mm.}$$

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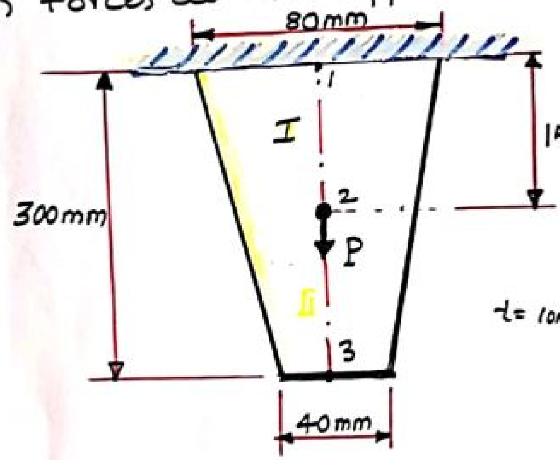
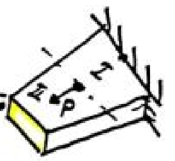
One dimensional problems

> For a tapered bar of Uniform thickness $t = 10\,mm$ as shown in fig. Find the displacements at the nodes by forming into two element model. The Bar has mass density $\rho = 7800\ Kg/m^3$, Young's modulus $E = 2\times10^5\ MN/m^2$ N/mm^2 lly. In addition to self weight, the bar is subjected to a Point load $P = 1\,KN$ at it's Centre. Also determine the reaction Forces at the Support.





Now for the taper bar,

The area at Node 1 $= 80 \times 10 = 800\ mm^2$.

Node 3 $= 40 \times 10 = 400\ mm^2$

& now area at node 2 $= \dfrac{800 + 400}{2} = 600\ mm^2$.

now for the stepped bar.

The Area of C/s for Element (I)

$A_1 = \dfrac{\text{Area at node1} + \text{area at node 2}}{2}$ (taper section I)

$= \dfrac{800 + 600}{2} = 700\ mm^2 = (70 \times 10)\,mm$.

Area of C/s for Element (II)

$A_2 = \dfrac{\text{Area of Node 2} + \text{Area of Node 3}}{2}$ (Taper section II)

$= \dfrac{600 + 400}{2} = 500\ mm^2$ $(50 \times 10)\,mm$.

The Formation for FE Equation.

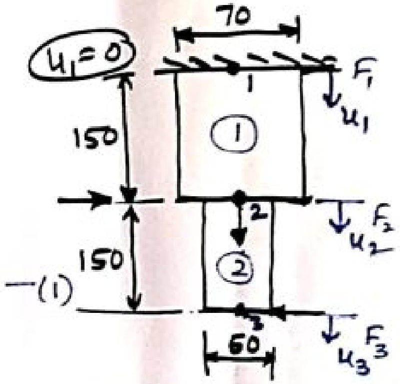
$[K][u] = [F]$

$\dfrac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = K$

$$K_1 = \frac{700 \times 2\times10^5}{150}\ \left(\frac{14\times10^6}{15}\right) \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ u_2 \end{array} \quad \text{— (i)}$$

$$K_2 = \frac{500 \times 2\times10^5}{150}\ \left(\frac{10\times10^6}{15}\right) \begin{array}{c} \begin{array}{cc} u_2 & u_3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} u_2 \\ u_3 \end{array} \quad \text{— (ii)}$$

$$[K] = \frac{10^6}{15} \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 14 & -14 & 0 \\ -14 & 24 & -10 \\ 0 & -10 & 10 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$



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The body for the Element 1

$W_1 = A_1 l_1 \rho g = 700 \times 150 \times 7800 \times 10^{-9} \times 9.81$

$W_1 = 8.034\ N$

Element 2

$W_2 = A_2 l_2 \rho g = 500 \times 150 \times 7800 \times 10^{-9} \times 9.81$

$= 5.739\ N$

$P = 1\ KN$ (Point load) at node 2 .

The nodal force at node 1 .

$$F_1 = \frac{\text{Body force of Element 1}}{2} = \frac{8.034}{2} = 4.017\ N$$

at node 2, $F_2 = \dfrac{\text{Body force of Elemnt 1}}{2} + \dfrac{\text{B.F of Elemnt 2}}{2} + P$

$$= 4.017 + \frac{5.739}{2} + 1000$$

$F_2 = 1000.887\ N$

at node 3, $F_3 = \dfrac{\text{Body force at Element 2}}{2} = \dfrac{5.739}{2} = 2.870N$

$$F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 4.017 \\ 1006.88 \\ 2.870 \end{bmatrix} \quad \text{—— (3)}$$

$$\frac{10^6}{15} \begin{array}{cc} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} & \\ \begin{bmatrix} \cancel{14} & \cancel{-14} & \cancel{0} \\ \cancel{-14} & 24 & -10 \\ \cancel{0} & -10 & 10 \end{bmatrix} & \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \end{array} \begin{bmatrix} \cancel{u_1} \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \cancel{4.017} \\ 1006.8 \\ 2.870 \end{bmatrix}$$

The Boundary Condition are,

$u_1 = 0$

$$\frac{10^6}{15} \begin{bmatrix} 24 & -10 \\ -10 & 10 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1006.887 \\ 2.870 \end{bmatrix}$$

$$\frac{10^6}{15} (24 u_2 - \cancel{10 u_3}) = 1006.887$$

$$\frac{10^6}{15} (-10 u_2 + \cancel{10 u_3}) = 2.870$$

$$\frac{10^6}{15} (14 u_2) = 1009.757 .$$

$$u_2 = \frac{1009.757 \times 15}{10^6}$$

$$\therefore \boxed{u_2 = 1082 \times 10^{-6}\ mm}$$

Substitute the Value of u_2 in (ii)

$$\frac{10^6}{15} (-10 \times 1082 \times 10^{-6} + 10 u_3) = 2.870 .$$

$$\therefore \boxed{u_3 = 1086 \times 10^{-6}\ mm}$$

The Reaction Force at node 1.

$$R_1 = \frac{10^6}{15} \begin{bmatrix} 14 & -14 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1082 \times 10^{-6} \\ 1086 \times 10^{-6} \end{bmatrix} = 4.017$$

$$R_1 = 10^6/15 (+0 - 14 \times 1082 \times 10^{-6} + 0) = 4.017$$

$$\frac{10^6}{15} (-14 \times 1082 \times 10^{-6}) = 4.017$$

$$\boxed{R_1 = -1013.8\ N}$$

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Stiffness Matrix For 2D(CST) Element :

The Stiffness matrix for Two Dimensional Element Can be formulated from the Expression.

Stiffness matrix $[K] = \int_V [B]^T [D][B]\, dV$. —(1)

[B] - Strain-Displacement matrix (Gradient) for two-Dimensional Element.

[D] - Stress-Strain relation matrix / Elasticity.

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}_{(6\times3)}$$

$$[D] = \frac{E}{(1-\mu^2)}\begin{vmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \left(\frac{1-\mu}{2}\right) \end{vmatrix} \rightarrow$$ For Plane Stress Condition.

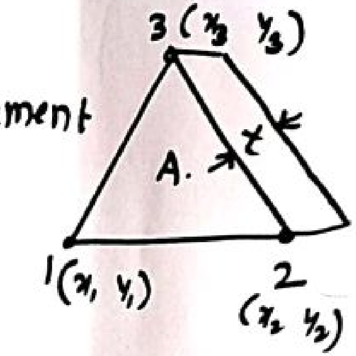
$$[D] = \frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix} (1-\mu) & \mu & 0 \\ \mu & (1-\mu) & 0 \\ 0 & 0 & \left(\frac{1-2\mu}{2}\right) \end{bmatrix}$$

For Plane Strain Condition.

Where, E - Young's modulus / Modulus of Elastr

μ - Poisson's ration.

A - Area of triangular Element



$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$[K] = [B]^T \cdot [D] \cdot [B] \cdot A\, t$.

Note (i) The value of t is the actual thickness of the body for Plane Stress.

(ii) Unity (1) for Plane Stress.

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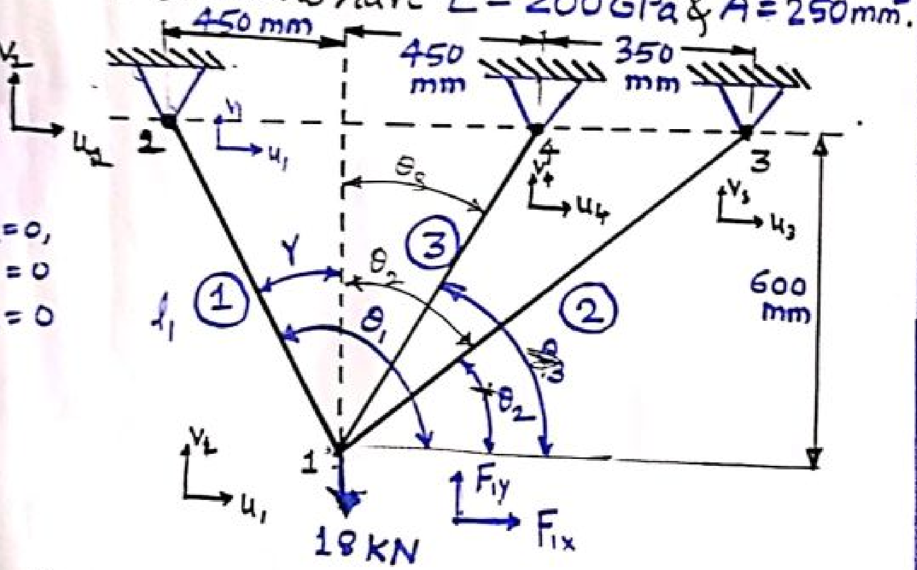
Analysis of Trusses

(7). For the Plane truss shown in fig. Determine the Horizontal & Vertical Displacements of the nodes & Stress in each Elements have $E = 200\,GPa$ & $A = 250\,mm^2$.



$V_2 = 0,$
$V_4 = 0$
$V_3 = 0$

$E = 200\,GPa = 200 \times 10^3\,MPa = 200 \times 10^3\,N/mm^2.$

$A_1 = 250\,mm^2.$

$$[K] = \frac{AE}{l}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \quad \text{— (I)}$$

$C - \cos\theta$
$S = \sin\theta.$

Nodal displacements:

(1) For Element (1)

$$\theta_1 = 90^\circ + \gamma$$
$$= 90 + \tan^{-1}\left(\frac{450}{600}\right)$$
$$\boxed{\theta_1 = 126.87^\circ}$$

$$l_1 = \sqrt{450^2 + 600^2} = 750\,mm$$

$$C = \cos\theta = \cos(126.87) = -0.6.$$

$$S = \sin\theta = \sin(126.87) = 0.8.$$

$$[K_1] = \frac{A_1E_1}{l_1} = \frac{250 \times 200 \times 10^3}{750} = 0.67 \times 10^5\,N/mm$$

The Stiffness matrix for (1)

$$[K_1] = 0.67 \times 10^5 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 0.36 & -0.48 & -0.36 & 0.48 \\ -0.48 & 0.64 & 0.48 & -0.64 \\ -0.36 & 0.48 & 0.36 & -0.48 \\ 0.48 & -0.64 & -0.48 & 0.64 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_2 \\ v_2 \end{array}.$$

For Element (2). $l = \sqrt{450^2 + 600^2}$

$$\theta_2 = \tan^{-1}\left(\frac{600}{800}\right) = 36.87^\circ.$$

$$\theta_3 = \tan^{-1}\left(\frac{600}{450}\right) = 53.13^\circ$$

For Element (2)

$$\theta_2 = 36.87^\circ.$$

$$l_2 = \sqrt{800^2 + 600^2} = 1000\,mm$$

$$C = \cos\theta = \cos(36.87^\circ) = 0.8.$$

$$S = \sin\theta = \sin(36.87) = 0.6.$$

$$[K_2] = \frac{A_2E_2}{l_2} = \frac{250 \times 200 \times 10^3}{1000} = 0.5 \times 10^5\,N/mm$$

The Stiffness matrix for (2).

$$[K_2] = 0.5 \times 10^3 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_3 \\ v_3 \end{array}$$

$||^{ly}$.

$$[K_3] = 0.67 \times 10^3 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48 \\ 0.48 & 0.64 & -0.48 & -0.64 \\ -0.36 & -0.48 & 0.36 & 0.48 \\ -0.48 & -0.64 & 0.48 & 0.64 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_4 \\ v_4 \end{array}$$

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$$[K] = 10^5 \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{cccccccc|c} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 & \\ \hline 0.8 & 0.24 & -0.24 & 0.32 & -0.32 & -0.24 & -0.24 & -0.32 & u_1 \\ 0.24 & 1.04 & 0.32 & -0.43 & -0.24 & -0.48 & -0.32 & -0.43 & v_1 \\ -0.24 & -0.32 & 0.24 & -0.32 & 0 & 0 & 0 & 0 & u_2 \\ 0.32 & -0.43 & -0.32 & 0.43 & 0 & 0 & 0 & 0 & v_2 \\ -0.32 & -0.24 & 0 & 0 & 0.32 & 0.24 & 0 & 0 & u_3 \\ -0.24 & -0.18 & 0 & 0 & 0.24 & 0.18 & 0 & 0 & v_3 \\ -0.24 & -0.32 & 0 & 0 & 0 & 0 & 0.24 & 0.32 & u_4 \\ -0.32 & -0.43 & 0 & 0 & 0 & 0 & 0.32 & 0.43 & v_4 \end{array} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{bmatrix}$$

$[K][u] = [F]$, Apply the B.C & force act on truss.

$u_2 = v_2 = 0, \quad u_3 = v_3 = 0, \quad u_4 = v_4 = 0,$

$F_{1y} = -18\,KN = -18000\,N, \quad F_{1x} = 0.$

$$10^5 \begin{bmatrix} 0.8 & 0.24 \\ 0.24 & 1.04 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -18000 \end{bmatrix}$$

$10^5(0.8\,u_1 + 0.24\,v_1) = 0$ — (i)

$10^5(0.24\,u_1 + 1.04\,v_1) = -18000$ — (ii)

$\rightarrow \boxed{u_1 = 0.056\,mm}$

$\rightarrow \boxed{v_1 = -0.187\,mm}$

The Stress at each ~~nodes~~ Element

$$\sigma_1 = \frac{E}{l}\begin{bmatrix} -c & -s & c & s \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$

$$= \frac{200 \times 10^3}{750} \begin{bmatrix} -0.6 & -0.8 & 0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 0.056 \\ -0.187 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\sigma_1 = 48.85\,N/mm^2}$$

HW $\sigma_2, \sigma_3 = ? = \sigma_2 = 13.48, \sigma_3 = 30.93\,N/$

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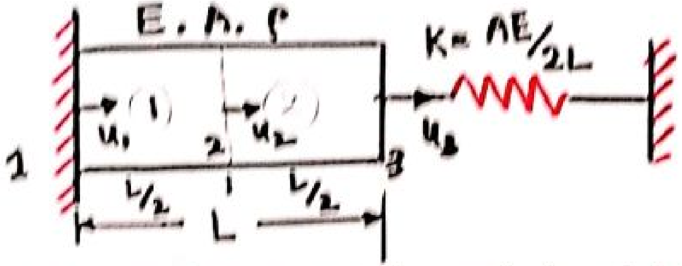
Dynamic Analysis :-

Determine the natural frequencies of the System shown in fig. (consider atleast 2).



Let, u_1, u_2 & u_3 → Axial Displacements at 1, 2 & 3

$A_1, E_1, l_1, K_1, \rho_1, m_1$ → Element (1)

$A_2, E_2, l_2, K_2, \rho_2, m_2$ → Element (2).

The Dynamic Equ^n of motion for Undamped free Vibration

$$\{[K] - \omega^2 [m]\}\{u\} = 0$$

Displacements.

$[K]$ → Stiffness matrix, ω^2 → Natural frequencies, $[m]$ → Mass matrix

$$\boxed{K = \frac{AE}{l}}$$

(i) SM for element (1). $[K] = \begin{array}{cc|c} 1 & 2 & \\ \end{array}$
$$[K] = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array}$$

SM for Element (2)
$$[K_2] = \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \end{array} \begin{array}{c} 2 \\ 3 \end{array}$$

The global Stiffness matrix

$$[K] = [K_1] + [K_2] = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} K_1 & -K_1 & 0 \\ -K_1 & (K_1+K_2) & -K_2 \\ 0 & -K_2 & K_2+K \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} (3\times3)$$

$$K_1 = \frac{A_1E_1}{l_1}, \quad K_2 = \frac{A_2E_2}{l_2}, \quad K = \frac{AE}{2l}$$

$$[K] = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{A_1E_1}{l_1} & -\frac{A_1E_1}{l_1} & 0 \\ -\frac{A_1E_1}{l_1} & \left(\frac{A_1E_1}{l_1} + \frac{A_2E_2}{l_2}\right) & -\frac{A_2E_2}{l_2} \\ 0 & -\frac{A_2E_2}{l_2} & \frac{A_2E_2}{l_2} + \frac{AE}{2l} \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

$$K_1 = \frac{AE}{(L/2)} = \frac{2AE}{L}$$

$$K_2 = \frac{AE}{L/2} = \frac{2AE}{L}$$

$$= \begin{bmatrix} \frac{2AE}{l} & -\frac{2AE}{l} & 0 \\ -\frac{AE}{l} & \left(\frac{2AE}{l}\right) & -\frac{AE}{l} \\ 0 & -\frac{AE}{l} & \frac{AE}{l} + \frac{AE}{2l} \end{bmatrix} = \frac{2AE}{l} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1.25 \end{bmatrix}$$

$$[K] = \frac{2AE}{l} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1.25 \end{bmatrix}$$

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Mass matrix for element ①

$$m_1 = \frac{\rho_1 A_1 l_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{\rho A . L/2}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$m_1 = \frac{\rho A L}{12}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \end{array}$$

for element ②

$$m_2 = \frac{\rho_2 A_2 l_2}{6}\begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array}\begin{array}{c} 2 \\ 3 \end{array}$$

$$= \frac{\rho A L/2}{6}\begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array}\begin{array}{c} 2 \\ 3 \end{array}$$

$$m_2 = \frac{\rho A L}{12}\begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array}\begin{array}{c} 2 \\ 3 \end{array}$$

The global mass matrix

$$[M] = [m_1 + m_2]$$

$$= \frac{\rho A L}{12}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

$$(K - \omega^2 M)\, u = 0$$

$$\left(\frac{2AE}{l}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1.25 \end{bmatrix} - \omega^2 \cdot \frac{\rho A L}{12}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 0$$

$$\div \frac{2AE}{l}.$$

$$\left\{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1.25 \end{bmatrix} - \left(\frac{\omega^2 \cdot \rho . L^2}{24E}\right)\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right\}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 0$$

$$\boxed{\frac{\omega^2 \cdot \rho . L^2}{24E} = \lambda}$$

$$\left\{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1.25 \end{bmatrix} - \lambda\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right\}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 0$$

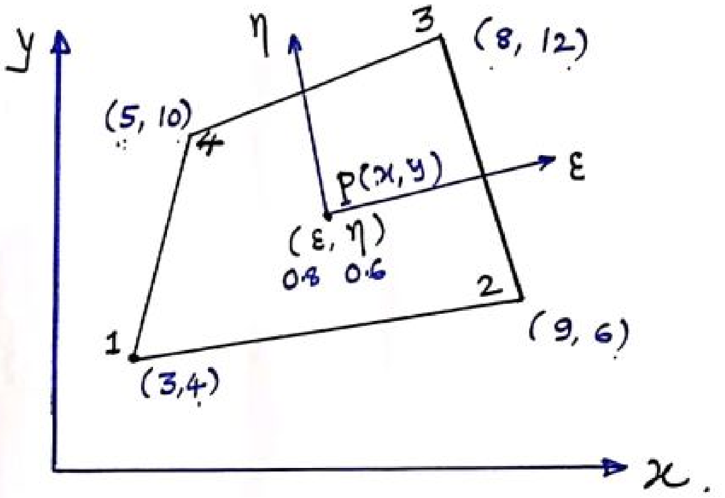
$$\begin{bmatrix} \cancel{(1-2\lambda)} & \cancel{(-1-\lambda)} & \cancel{0} \\ \cancel{(-1-\lambda)} & (2-4\lambda) & (-1-\lambda) \\ \cancel{0} & (-1-\lambda) & (1.25-2\lambda) \end{bmatrix}\begin{Bmatrix} \cancel{u_1} \\ u_2 \\ u_3 \end{Bmatrix} = 0$$

$$\begin{bmatrix} (2-4\lambda) & (-1-\lambda) \\ (-1-\lambda) & (1.25-2\lambda) \end{bmatrix}\boxed{\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}} = 0$$

$$\boxed{\lambda = 1.421}$$

Iso parametric Elements

Determine the Cartesian Co-ordinates of the Point 'P' which has local Co-ordinates $\varepsilon = 0.8$ and $\eta = 0.6$ as Shown in fig.



→ Global — Cartisian Co-ordinates
→ Local — Natural "

Now Cartisian (global) Co-ordinates of the Element at nodes 1, 2, 3 & 4.

$x_1 = 3$	$y_1 = 4$
$x_2 = 9$	$y_2 = 6$
$x_3 = 8$	$y_3 = 12$
$x_4 = 5$	$y_4 = 10$

& the local Coordinate System of the point P is $\varepsilon = 0.8$ & $\eta = 0.6$.

For, Cartisian coordinate System,

→ $x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$

→ $y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$

Where N_1, N_2, N_3 & N_4 are the shape function which are given by

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta)$$
$$N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta)$$
$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta)$$
$$N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta).$$

Substituting the Value of $\varepsilon = 0.8$ & $\eta = 0.6$.

$$N_1 = \frac{1}{4}(1-0.8)(1-0.6) = 0.02$$
$$N_2 = \frac{1}{4}(1+0.8)(1-0.6) = 0.18$$
$$N_3 = \frac{1}{4}(1+0.8)(1+0.6) = 0.72$$
$$N_4 = \frac{1}{4}(1-0.8)(1+0.6) = 0.08$$
$$= 1$$

Shape function the Global Co-ordination

$$x = (0.02)(3) + 0.18(9) + (0.72)(8) + 0.08 \times 5 = 7.84$$
$$y = (0.02)(4) + 0.18(6) + (0.72)(12) + 0.08 \times 10 = 10.6.$$

→ P(7.84, 10.6).

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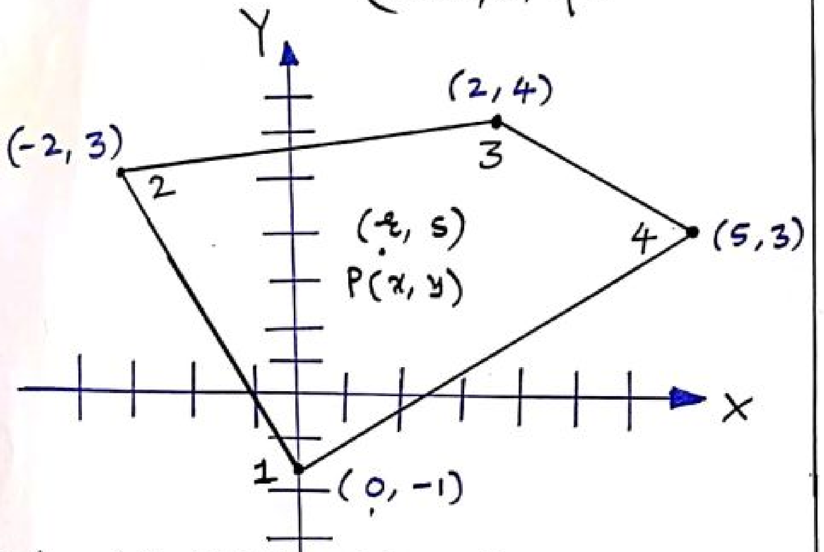
Isoparametric Elements

The Cartisian Co-ordinates of the Corner nodes of a quadrilateral element are given by (0, −1), (−2, 3), (2, 4) and (5, 3). Find the Coordinates transformation between the global & local Coordinates. Using this, Determine the Cartisian Co-ordinates of the point defined by $(r, s) = (0.5, 0.5)$ in the local Co-ordinates System.

(zeeta, ε, η.)



Global → (Cartisian) P(x, y)

Local → (r, s) = (0.5, 0.5)

$x_1 = 0$	$y_1 = -1$
$x_2 = -2$	$y_2 = 3$
$x_3 = 2$	$y_3 = 4$
$x_4 = 5$	$y_4 = 3$

& the local Co-ordinates are $r = 0.5$ & $s = 0.5$.

The transformation bet^n global & local Cordinates is

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4 \quad \text{— (i)}$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4 \quad \text{— (ii)}$$

The Shape for the node 1,

$$N_1 = \frac{1}{4}(1-r)(1-s) = \frac{1}{4}(1-0.5)(1-0.5) = 0.0625$$

$$N_2 = \frac{1}{4}(1+r)(1-s) = \frac{1}{4}(1+0.5)(1-0.5) = 0.1875$$

$$N_3 = \frac{1}{4}(1+r)(1+s) = \frac{1}{4}(1+0.5)(1+0.5) = 0.5625$$

$$N_4 = \frac{1}{4}(1-r)(1+s) = \frac{1}{4}(1-0.5)(1+0.5) = 0.1875$$

$$\boxed{N_1 + N_2 + N_3 + N_4 = 1}$$

Substitute the Shape function & Global Coordinates in Equation (i) & (ii).

$$x = (0.0625)(0) + (0.1875)(-2) + (0.5625)(2) + (0.1875)(5)$$

$$= 0 - 0.375 + 1.125 + 0.9375$$

$$\boxed{x = 1.6875}$$

$$y = (0.0625)(-1) + (0.1875)(3) + (0.5625)(4) + (0.1875)(3)$$

$$= -0.0625 + 0.5625 + 2.25 + 0.5625$$

$$\boxed{y = 3.3125}$$

1

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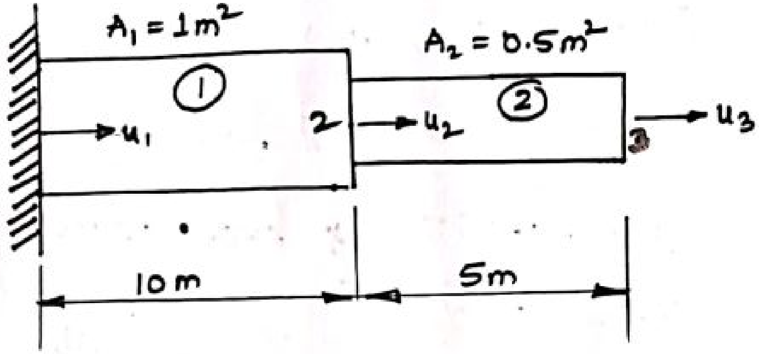
Dynamic Analysis :

Determine the eigen Values & Frequencies for the Stepped bar shown in fig.



$E = 30 \times 10^{10}\ N/m^2$

Sp. Weight $= 8500\ kg/m^3$

For Element ①	For Element ②
$A_1 = 1m^2$	$A_2 = 0.5m^2$
$L_1 = 10m$	$L_2 = 5m$
$E_1 = 30\times10^{10} N/m^2$	$E_2 = 30\times10^{10} N/m^2$
$\rho_1 = 8500\ Kg/m^3$	$\rho_2 = 8500\ Kg/m^3$

$$[A - \lambda B]x = 0$$

$$\boxed{[[K] - \omega^2[M]]x = 0} \rightarrow \text{UnDamped Vibration Eq}^n$$

For the Element ①

$$\begin{array}{c} \\ [K_1] = \frac{A_1E_1}{l_1} \end{array} \begin{array}{cc} 1 & 2 \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array} = \frac{1\times \cancel{30}\times10^{10}}{\cancel{10}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 3\times10^{10}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Element mass matrix

$$[M_1] = \frac{\rho_1A_1L_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{8500\times1\times10}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{85\times10^3}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

For Element ②

$$[K_2] = \frac{A_2E_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{0.5\times30\times10^{10}}{5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 3\times10^{10}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix};$$

$$\text{lly } [M_2] = \frac{\rho_2A_2L_2}{6}\begin{bmatrix} 2 & +1 \\ +1 & 2 \end{bmatrix} = \frac{8500\times0.5\times5}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{85\times10^3}{24}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix};$$

→ [K] - Global Stiffness , [M] - Global Mass Matrix.

$$[K] = 3\times10^{10}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad [M] = \frac{85\times10^3}{24}\begin{bmatrix} 8 & 4 & 0 \\ 4 & 10 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\boxed{\{[K] - \omega^2 M\}x = 0}$$

$$\left[\frac{3\times10^{10}}{1}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \omega^2\times\frac{85\times10^3}{24}\begin{bmatrix} 8 & 4 & 0 \\ 4 & 10 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 0$$

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$$\left[\cancel{3\times10}^{10} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \underbrace{1.181\times10^{-7}\omega^2}_{\lambda} \begin{bmatrix} 8 & 4 & 0 \\ 4 & 10 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 0$$

...et. $\boxed{\lambda = 1.181\times10^{-7}\omega^2}$

$$\left[\cancel{3\times10}^{10} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 8 & 4 & 0 \\ 4 & 10 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 0$$

$$\cancel{3\times10}^{10} \rightarrow \begin{bmatrix} \cancel{(1-8\lambda)} & \cancel{(-1-4\lambda)} & \cancel{0} \\ \cancel{(-1-4\lambda)} & (2-10\lambda) & (-1-\lambda) \\ \cancel{0} & (-1-\lambda) & (1-2\lambda) \end{bmatrix} \begin{bmatrix} \cancel{u_1} \\ u_2 \\ u_3 \end{bmatrix} = 0$$

$\boxed{u_1 = 0}$

$$\Rightarrow \begin{bmatrix} (2-10\lambda) & (-1-\lambda) \\ (-1-\lambda) & (1-2\lambda) \end{bmatrix} \times \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = 0$$

To find the non-zero value of circular freq. equate the det. of co-eff. matrix of equ^n to zero

$$\begin{vmatrix} (2-10\lambda) & (-1-\lambda) \\ (-1-\lambda) & (1-2\lambda) \end{vmatrix} = 0$$

$$(2-10\lambda)(1-2\lambda) - (-1-\lambda)(-1-\lambda) = 0$$

$$2 - \underbrace{4\lambda - 10\lambda} + 20\lambda^2 - \left[1 + \underbrace{\lambda + \lambda} + \lambda^2\right] = 0$$

$$20\lambda^2 - 14\lambda + 2 - 1 - 2\lambda - \lambda^2 = 0$$

$$\boxed{19\lambda^2 - 16\lambda + 1 = 0}$$

$\therefore x = -b \pm \sqrt{b^2 - 4ac} / 2a$

$\lambda = +0.0679$ & $\cancel{0.067}$ 0.7742

$$\boxed{\begin{aligned} \lambda_1 &= 0.0679 \\ \lambda_2 &= 0.7742 \end{aligned}}$$ Eigen Value.

The frequency are.

$$\boxed{\lambda = 1.181\times10^{-7}\omega^2}$$

$\rightarrow 0.0679 = 1.181\times10^{-7}\,\omega_1^2$

$$\omega_1^2 = \frac{0.0679}{1.181\times10^{-7}} = 0.0575\times10^{7}$$

$$\boxed{\omega_1 = 57.5\times10^{4}} = 0.0575\times10^{7}\ \text{rad/s}$$

$\rightarrow 0.7747 = 1.181\times10^{-7}\,\omega_2^2$

$$\omega_2^2 = \frac{0.7747}{1.181\times10^{-7}} = 0.6555\times10^{7}$$

$\omega_2 = 2560.3$ rad/s.

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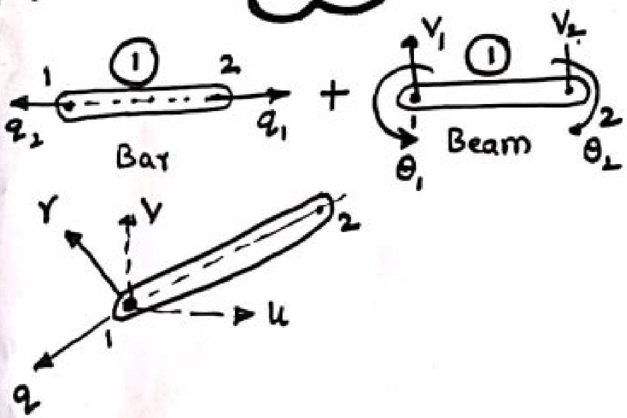
Stiffness Matrix for a Frame element



q - axial Displacement
r - Radial Displacement
u - Horizontal "
v - Vertical "

The Stiffness matrix for Bar Element,

$$[K_{br}] = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{array}{cc} u_1 & u_2 \\ \left[\begin{matrix} \frac{AE}{l} & -\frac{AE}{l} \\ -\frac{AE}{l} & \frac{AE}{l} \end{matrix}\right] & \begin{matrix} u_1 \\ u_2 \end{matrix} \end{array} \quad -(1)$$

Simpl Similarly the Stiffness matrix for Beam Element,

$$[K_{bm}] = \frac{EI}{l^3}\begin{array}{cc} \begin{matrix} v_1 & \theta_1 & v_2 & \theta_2 \end{matrix} & \\ \left[\begin{matrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{matrix}\right] & \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{matrix} \end{array} \quad -(2)$$

The Combined Stiffness matrix for the Frame elements

$$[K_F] = [K_{br}] + [K_{bm}]$$

now, assume, $G = \frac{AE}{l}$, $H = \frac{EI}{l^3}$

then Eq^n(1), $[K_{br}] = \begin{array}{cc} \begin{matrix} u_1 & u_2 \end{matrix} & \\ \left[\begin{matrix} G & -G \\ -G & G \end{matrix}\right] & \begin{matrix} u_1 \\ u_2 \end{matrix} \end{array} \quad -(3)$

for Equ^n(2), $[K_{bm}] = \begin{array}{cc} \begin{matrix} u_1 & \theta_1 & v_2 & \theta_2 \end{matrix} & \\ \left[\begin{matrix} 12H & 6Hl & -12H & 6Hl \\ 6Hl & 4Hl^2 & -6Hl & 2Hl^2 \\ -12H & -6Hl & 12H & -6Hl \\ 6Hl & 2Hl^2 & -6Hl & 4Hl^2 \end{matrix}\right] & \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{matrix} \end{array} \quad -(4)$

now the Equ^n is,

$$[K] = \begin{array}{cc} \begin{matrix} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{matrix} & \\ \left[\begin{matrix} G & 0 & 0 & -G & 0 & 0 \\ 0 & 12H & 6Hl & 0 & -12H & 6Hl \\ 0 & 6Hl & 4Hl^2 & 0 & -6Hl & 2Hl^2 \\ -G & 0 & 0 & G & 0 & 0 \\ 0 & -12H & -6Hl & 0 & 12H & -6Hl \\ 0 & 6Hl & 2Hl^2 & 0 & -6Hl & 4Hl^2 \end{matrix}\right] & \begin{matrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{matrix} \end{array}$$

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Design of Cylinder

1. Bore & length of the Cylinder

The bore (i.e inner dia) & length of the Cylinder determined as:

Let, P_m – Indicated mean eff. pressure, N/mm^2

D – Cylinder bore in mm,

A – Area of c/s of the Cylinder, $\frac{\pi}{4} D^2$

L – length of stroke in metre.

N – Speed of the Engine, rpm.

n – No. of working stroke per min.

($n = N$, For 2 Stroke

$n = \frac{N}{2}$ For 4 Stroke.

W.K.T, The power produced inside the Cylinder, i.e Indicated power,

$$I.P = \frac{P_m . L . A . n}{60} \text{ Watt.}$$

From this Expression, the bore (D) & length of the Stroke (l) is determined.

1 > The length of stroke is generally taken as 1.25D to 2D

> Since there is a clearance on both side of the cylinder,

∴ length of the cylinder is taken as $L = 1.15 \times$ length of stroke $= 1.15 \times l$.

Note : a). $IP = \frac{BP}{\eta_{mech}} \Rightarrow \eta_{mec} = \frac{BP}{IP}$

b). Max. gas pressure (P_{max}) may be taken as 9 to 10 times the (P_m).

$$\boxed{P_{max} = 10 \times P_m}$$

2. Thickness of the Cylinder Wall :

a). longitudinal Stress & b) Circumferencial Stress.

> longitudinal Stress, $\sigma_l = \frac{F}{A} = \frac{\frac{\pi}{4} \times D^2 \times p}{\frac{\pi}{4}(D_o^2 - D^2)} = \frac{D^2 p}{D_o^2 - D^2}$

> Circumferential Stress, $\sigma_c = \frac{F}{A} = \frac{D \times l \times p}{2t \times l} = \frac{D.p}{2t}$.

Net longitudinal Stress $= \sigma_l - \frac{\sigma_c}{m}$

Net Circumferential Stress $= \sigma_c - \frac{\sigma_l}{m}$

∴ The thickness of the cylindrical wall

$$t = \frac{p.D}{2\sigma_c} + C \quad , \quad \sigma_c = 35 \text{ to } 100 \text{ MPa}$$

C = table (All. for reboring

3. Cylinder Flange & Studs :.

4. Cylinder Head : $t_h = D\sqrt{\frac{C.P}{\sigma_c}}$, $\sigma_c = 30$ to 50 MPa.

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A Four Stroke diesel Engine has the following Specification: Brake power = 5KW, Speed = 1200 rpm Indicated mean effective pressure = 0.35 N/mm²; Mechanical efficiency = 80%.

Determine: Bore & length of the Cylinder,
Thickness of the Cylinder head,
Size of the Studs of the Cylinder head.

$BP = 5\,KW = 5\times10^3\,W = 5000\,W$

$N = 1200\ r.p.m \Rightarrow \eta = \frac{N}{2} = \frac{1200}{2} = 600$

$P_m = 0.35\ N/mm^2$.

$\eta_{mech} = 80\% = \frac{80}{100} = 0.8$.

1. Bore & length of the Cylinder:

$IP = \frac{P_m \times L \cdot A \cdot n}{60}$ Watt

$\eta_{mech} = \frac{BP}{IP} \Rightarrow \frac{5000}{0.8} = IP$

$\boxed{IP = 6250\ W.}$

$A = \frac{\pi}{4} \cdot (D)^2 =$

$6250 = \frac{0.35 \times l \times \pi \cdot D^2 \times 600}{4}$

$l = 1.5D$

$6250 = \frac{0.35 \times 1.5D \times \pi D^2 \times 600}{60}$

$D^3 = \frac{6250}{4.12\times10^{-3}} = 1517\times10^3$.

$\boxed{D = 115\ mm}$, $\boxed{l = 1.5\times115 = 172.5\,mm}$

Taking clearance on both Side of the Cylinder

$L = 1.15 \times l = 1.15 \times 172.5 = 198\ mm$.

$\boxed{L = 200\ mm}$

2. Thickness of the Cylinder head: (t_h):

$(t_h) = D\sqrt{\frac{Cp}{\sigma_t}} = 115\sqrt{\frac{0.1\times3.15}{42}} = 9.96$.

$\rightarrow p = 10\times p_m$
$= 10\times0.35$
$= 3.15\ N/mm^2$

$\boxed{t_h = 10\ mm}$

3. Side of the Studs of the Cylinder head:

WKT, the forces acting on the Cylinder head

$= \pi/4 \times D^2 \times p = \frac{\pi}{4}(115)^2 \times 3.15$

$= 32702\ N$

The no. of Studs are taken between $0.01D + 4\ (n_s)$

$= 0.01\times115 + 4 = 5.15$ to

$0.02D + 4 = 0.02\times115 + 4 = 6.3\ [5.15$ to $6.3]$

$\boxed{n_s = 6}$

$\rightarrow d_c = 0.84d$

$\frac{\pi}{4} D^2 p = \pi/4\,(d_c)^2 \cdot \sigma_t$. $\rightarrow \sigma_t \rightarrow 65\,MPa$.

$32702 = \frac{\pi}{4}(0.84d)^2 \times 65 \Rightarrow \boxed{d = 12.3 \simeq 14\ mm}$

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Strain - Displacement Relationship Matrix

[B]. for CST,

Plane Stress: When the Elastic body is very thin & there are no loads applied

$$[D] = \frac{E}{(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \left(\frac{1-\mu}{2}\right) \end{bmatrix} \quad \text{—(i)}$$

Plane Strain: The members that are not free to Expand in the direction $\perp^r$ to the Plane of applied load.

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} (1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \left(\frac{1-2\mu}{2}\right) \end{bmatrix} \quad \text{—(ii)}$$

Where, the material properties like

μ - Poisson's ratio

E - Elasticity of Modulus. N/m^2.

$$\delta(x,y) = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

$$u = N_1u_1 + N_2u_2 + N_3u_3$$

$$v = N_1v_1 + N_2v_2 + N_3v_3.$$

The Strain which is the ratio of displacement to length can be X & Y axes displacement. (u & v.)

The normal Strain in x-direction

$$e_x = \frac{\partial u}{\partial x} = \frac{\partial N_1}{\partial x} \cdot u_1 + \frac{\partial N_2}{\partial x} \cdot u_2 + \frac{\partial N_3}{\partial x} \cdot u_3 \quad \text{—(iii)}$$

The Strain in Y-direction,

$$e_y = \frac{\partial v}{\partial y} = \frac{\partial N_1}{\partial y} \cdot v_1 + \frac{\partial N_2}{\partial y} \cdot v_2 + \frac{\partial N_3}{\partial y} \cdot v_3 \quad \text{—(iv)}.$$

The Shear Strain in X-Y plane

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial N_1}{\partial y} u_1 + \frac{\partial N_2}{\partial y} u_2 + \frac{\partial N_3}{\partial y} u_3 + \frac{\partial N_1}{\partial x_2} v_1 + \frac{\partial N_2}{\partial x} v_2 + \frac{\partial N_3}{\partial x} v_3 \quad \text{—(v)}.$$

Writing the equ^n, in matrix form,

$$\begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad \text{—(vi)}$$

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Where, N_1, N_2, N_3 are the shape functions of x & y & $u_1, v_1, u_2, v_2, u_3, v_3$ are the constant value.

$$N_1 = \frac{1}{2A}(\alpha_1 + \beta_1 x + \gamma_1 y).$$

$$N_2 = \frac{1}{2A}(\alpha_2 + \beta_2 x + \gamma_2 y)$$

$$N_3 = \frac{1}{2A}(\alpha_3 + \beta_3 x + \gamma_3 y).$$

now,

$$\frac{\partial N_1}{\partial x} = \frac{\beta_1}{2A}, \quad \frac{\partial N_2}{\partial x} = \frac{\beta_2}{2A}, \quad \frac{\partial N_3}{\partial x} = \frac{\beta_3}{2A}$$

$$\frac{\partial N_2}{\partial y} = \frac{\gamma_1}{2A}, \quad \frac{\partial N_2}{\partial y} = \frac{\gamma_2}{2A} \quad \& \quad \frac{\partial N_3}{\partial y} = \frac{\gamma_3}{2A}.$$

Substitute the above values in Eqn,

$$\begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

The above equation can be written as,

$$e = [B].\{\delta\}$$

i.e. $[B]$ = Strain-displacement relationship matrix for CST is,

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

Where,

$\beta_1 = (y_2 - y_3)$, $\beta_2 = (y_3 - y_1)$ & $\beta_3 = (y_1 - y_2)$

$\gamma_1 = (x_3 - x_2)$, $\gamma_2 = (x_1 - x_3)$ & $\gamma_3 = (x_2 - x_1)$

$[\delta]$ — Nodal displacement Vector

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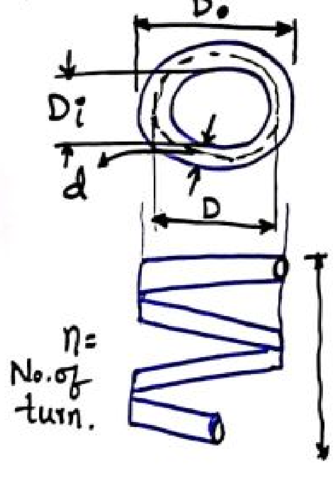
Design of Helical Spring

$C = \frac{D}{d} = 5 \Rightarrow \boxed{5d = D}$

Design a helical Compression Spring for a Maximum load of 1000 N for a deflection of 25 mm using the Value of Spring Index as 5.

The Maximum permissible Shear Stress for Spring wire is 420 MPa and Modulus of Rigidity is 84 KN/mm². Take Wahl's factor, $K = \frac{4C-1}{4C-4} + \frac{0.615}{C}$

Where C = Spring index.



$W = F = 1000\ N$

$\delta = 25\ mm$

$C = \frac{D}{d} = 5.$

$\tau = 420\ MP = 420\ N/mm^2.$

$G = 84\ KN/mm^2 = 84 \times 10^3\ N/mm^2.$

$K = \frac{4C-1}{4C-4} + \frac{0.615}{C}$

i). Mean dia. of the Spring Coil.
ii). Number of turn of the coil :
iii). Free length of the Spring
iv). Pitch of the coil.

1. **Mean dia. of the Spring Coil:**

(11.2) W.K.T, $K = \frac{4C-1}{4C-4} + \frac{0.615}{C} = \frac{4\times5-1}{4\times5-4} + \frac{0.615}{5} = 1.31.$

& the maximum Shear Stress (τ)

$$\tau = \frac{8FDK}{\pi d^3} = \frac{8\times1000\times5d\times1.31}{\pi d^3} = \frac{8\times1000\times5\times1.31}{\pi d^2}$$

$$\tau = \frac{16677}{d^2} \Rightarrow 420 = \frac{16677}{d^2} \Rightarrow \boxed{d = 6.3\ mm}$$

According SWG, $\boxed{d = 6.401\ mm}$

∴ Mean dia. of the spring coil, $D = Cd = 5\times6.401$

$\boxed{D = 32.005\ mm}$

& Outer dia. of the Spring coil, $D_o = D + d = 32.005 + 6.401$

$\boxed{D_o = 38.406\ mm}$

2. **Number of turn of the coil (n):**

WKT, the compression of the Spring (δ).

$$\delta = \frac{8FD^3 \cdot n}{G \cdot d^4} = \frac{8F \cdot C^3 \cdot n}{Gd} = \frac{8\times1000\times5^3\times n}{84\times10^3\times6.401}$$

$$25 = \frac{8\times1000\times5^3\times n}{84\times10^3\times6.401} = 1.86\,n.$$

$$n = \frac{25}{1.86} = 13.44. \text{ Say } 14.$$

3. **Free length:** $n'd + \delta + 0.15\delta = (n+2)d + \delta + 0.15\delta$
$= 16d + 25 + 0.15\times25 = 131.2\ mm.$

4. **Pitch of the Coil:** $= \frac{\text{Free length}}{n'-1} = \frac{131.2}{16-1} = 8.75\ mm.$

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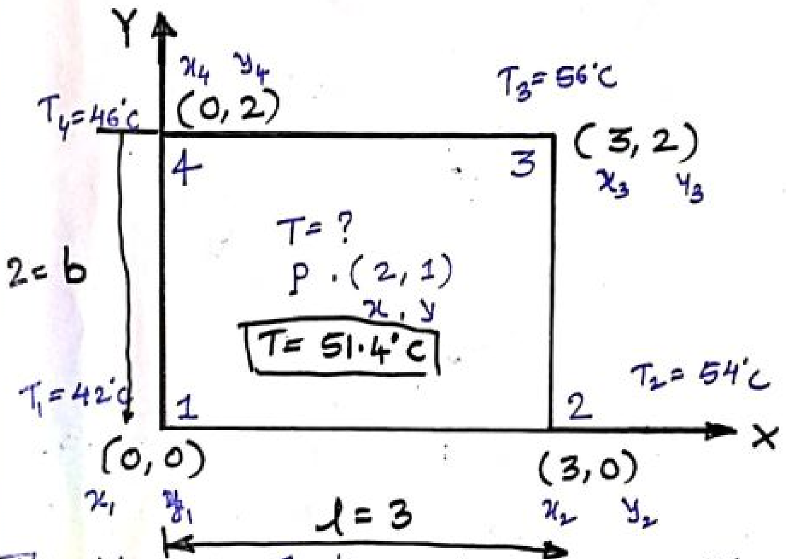
2D-Scalar Variable problems

For a Four noded rectangular Element Shown in fig. Determine the temperature at the Point (2, 1). The nodal temperature are $T_1 = 42°c$, $T_2 = 54°c$, $T_3 = 56°c$ & $T_4 = 46°c$.



The Variation of temperature T on the Element Can be Expressed as

$$T = N_1T_1 + N_2T_2 + N_3T_3 + N_4T_4$$

N – Shape function

T – Temp. at each nodes.

Now the shape functions are

$$N_1 = \left(1 - \frac{x}{l}\right)\left(1 - \frac{y}{b}\right)$$

$$N_2 = \left(\frac{x}{l}\right)\left(1 - \frac{y}{b}\right)$$

$$N_3 = \frac{xy}{lb}$$

$$N_4 = \left(1 - \frac{x}{l}\right)\left(\frac{y}{b}\right)$$

$l = x_2 - x_1 = 3 - 0 = 3$

$b = y_4 - y_1 = 2 - 0 = 2$

& the nodal temperatures are,

$T_1 = 42°c$, $T_2 = 54°c$, $T_3 = 56°c$ & $T_4 = 46°c$

$$N_1 = \left[1 - \frac{2}{3}\right]\left[1 - \frac{1}{2}\right] = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

$$N_2 = \frac{2}{3} \times \left(1 - \frac{1}{2}\right) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$$

$$N_3 = \frac{2 \times 1}{3 \times 2} = \frac{1}{3}$$

$$N_4 = \left(1 - \frac{2}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

now, the temp. at the Point P(2, 1) is given by,

$$T = N_1T_1 + N_2T_2 + N_3T_3 + N_4T_4$$

$$= \left[\frac{1}{6} \times 42\right] + \left[\frac{1}{3} \times 54\right] + \left[\frac{1}{3} \times 56\right] + \left[\frac{1}{6} \times 46\right] = 7 + 18 + 18.7 + 7.7$$

$$51.4°$$ Ans.

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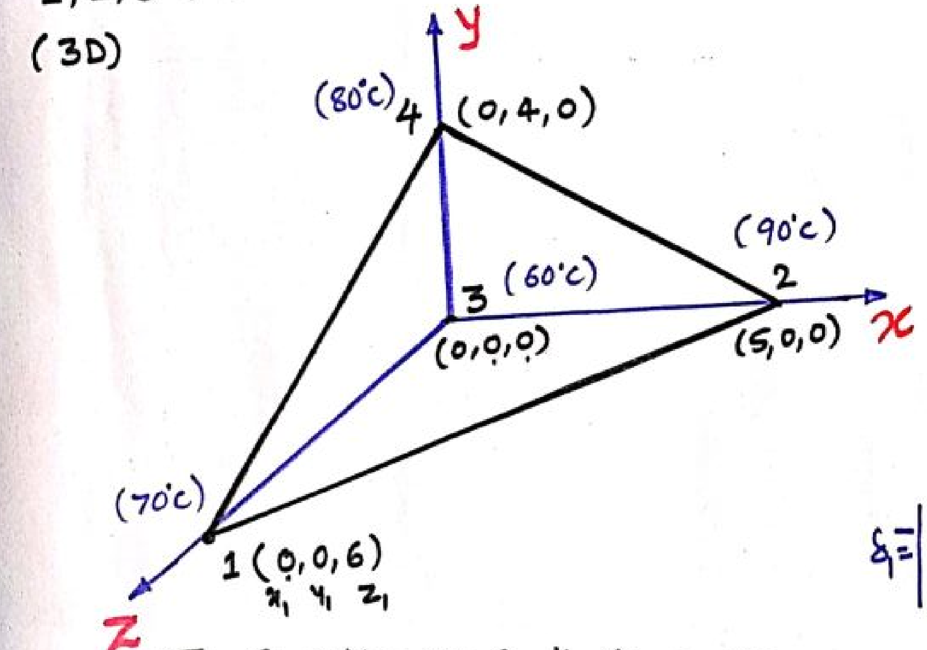
FEM-Heat transfer

> For a Four noded tetrahedral Solid Element Shown in fig. Determine the temperature at Point P(2,1,3), Given that the temperatures at nodes 1,2,3 & 4 are 70°c, 90°c, 60°c, 80°c respectively.

(3D)



The Co-ordinates & the temp. are,

At Node 1, $x_1 = 0$, $y_1 = 0$, $z_1 = 6$, $T_1 = 70°c$

At Node 2, $x_2 = 5$, $y_2 = 0$, $z_2 = 0$, $T_2 = 90°c$

At Node 3, $x_3 = 0$, $y_3 = 0$, $z_3 = 0$, $T_3 = 60°c$

Node 4, $x_4 = 0$, $y_4 = 4$, $z_4 = 0$, $T_4 = 80°c$

'T' at any point, the nodal Values.

$$\boxed{T(x,y,z) = N_1T_1 + N_2T_2 + N_3T_3 + N_4T_4}$$

Where, N_1, N_2, N_3 & N_4 are Shape function, 1,2,3 & 4:-

$N_1 = \frac{1}{6V}(\alpha_1 + \beta_1 x + \gamma_1 y + \delta_1 z)$, $N_3 = \frac{1}{6V}(\alpha_3 + \beta_3 x + \gamma_3 y + \delta_3 z)$

$N_2 = \frac{6}{6V}(\alpha_2 + \beta_2 x + \gamma_2 y + \delta_2 z)$, $N_4 = \frac{1}{6V}(\alpha_4 + \beta_4 x + \gamma_4 y + \delta_4 z)$

(V) - Volume of the tetrahedral Element.

$$6V = \begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix}_{(4\times4)} = \begin{vmatrix} 1 & 0 & 0 & 6 \\ 1 & 5 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 4 & 0 \end{vmatrix}_{(4\times4)} \quad \begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

$$= 1\begin{vmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{vmatrix} - 0(0) + 0 - 6\begin{vmatrix} 1 & 5 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 4 \end{vmatrix}$$

$$= 1\left[5(0-0)\right] + 0 + 0 - 6\left[1(0) - 5(4-0)\right]$$

$$6V = 0 + 0 + 0 - 6\left[0 - 20\right] = +120.$$

$\alpha_1 = \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}$, $\beta_1 = \begin{vmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix}$, $\gamma_1 = \begin{vmatrix} 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{vmatrix}$, $\delta_1 = -\begin{vmatrix} 1 & x_2 & z_1 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{vmatrix}$

$\alpha_2 = -\begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix}$, $\beta_2 = \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix}$, $\gamma_2 = -\begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{vmatrix}$, $\delta_2 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{vmatrix}$

$\alpha_3 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{vmatrix}$, $\beta_3 = -\begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_4 & z_4 \end{vmatrix}$, $\gamma_3 = \begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_4 & z_4 \end{vmatrix}$, $\delta_3 = -\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_4 & y_4 \end{vmatrix}$

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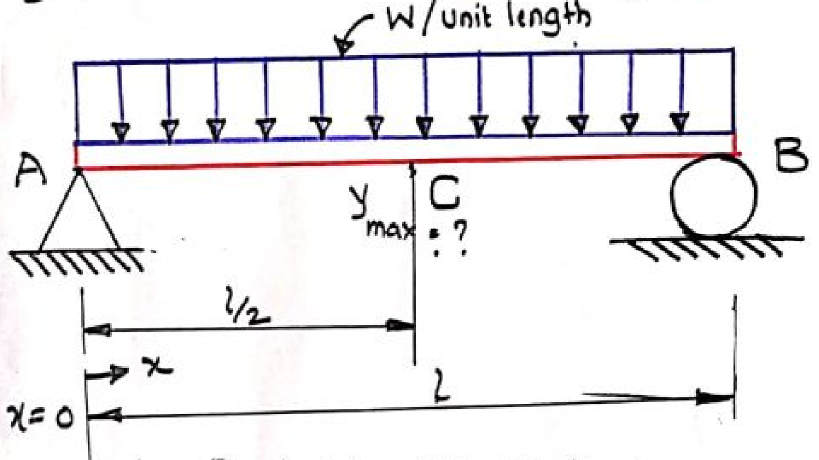
Rayleigh-Ritz Method

Determine the deflection at the Centre of a Simply Supported beam of Span length 'l' Subjected to Uniformly distributed load throught its length as shown in fig. (Use Rayleigh Ritz Method).



According to Rayleigh - Ritz Method,
When a displacement function make the total Potential Energy minimum, that function may be Considered as Approximate Solution.

Consider a displacement fluction, (i.e deflection) as.

$$y = a_1 + a_2 x + a_3 x^2 \quad -(I)$$

The Boundary Conditions are $y = 0$, at $x = 0$ & $x = l$.

Equ^n (I), $\boxed{a_1 = 0}$

$$0 = \cancel{a_1} + a_2 l + a_3 l^2 \Rightarrow$$

$$a_2 l + a_3 l^2 = 0 \Rightarrow \quad a_2 l = -a_3 l^2$$

$$\boxed{a_2 = -a_3 l}$$

Then the deflection y can be Written as.

$$y = 0 + (-a_3 l)x + a_3 x^2.$$

$$= -a_3 l x + a_3 x^2 = a_3(x^2 - lx) \quad -(2)$$

Diff. two times, We get,

$$\frac{dy}{dx} = a_3(2x - l)$$

$$\frac{d^2y}{dx^2} = a_3 2 = 2a_3$$

Strain Energy.

$$U = \frac{EI}{2}\int_0^l \left[\frac{d^2y}{dx^2}\right]^2 . dx.$$

$$= \frac{EI}{2}\int_0^l (2a_3)^2 . dx = \cancel{4}^{2} a_3^2 \frac{EI}{\cancel{2}} =$$

$$\boxed{U = 2a_3^2 EI . l}$$

Work done, $W = \int_0^l w.y.dx = \int_0^l w\, a_3(x^2 - lx).\, dx$

$$W = wa_3\left[\frac{x^3}{3} - l.\frac{x^2}{2}\right]_0^l = wa_3\left[\frac{l^3}{3} - \frac{l^3}{2}\right] = \frac{-wa_3 l^3}{6}.$$

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Total potential Energy

π = Strain Energy − W. done

$$\boxed{\pi = U - W}$$

$$\pi = 2a_3^2 EI.l + \frac{Wa_3l^3}{6}$$

$$\frac{\partial \pi}{\partial a_3} = 4a_3.EIl + \frac{Wl^3}{6}$$

$$\frac{\partial \pi}{\partial a_3} = 0$$

$$4a_3EIl + \frac{Wl^3}{6} = 0.$$

$$\therefore a_3 = -\frac{W\cancel{l^3}^{\,2}}{6} \times \frac{1}{4EI.\cancel{l}}$$

$$\boxed{a_3 = -\frac{Wl^2}{24.EI}}$$

Then the deflection,

$$y = a_3(x^2 - lx).$$

$$= \frac{-Wl^2}{24EI}(x^2 - lx) \quad \text{—(3)}$$

y_{max} = ?

Maximum deflection at $x = l/2$.

$$y_{max} = \frac{-W.l^2}{24EI}\left[\frac{l^2}{4} - l.\frac{l}{2}\right]$$

$$= \frac{-Wl^2}{24EI}\left[\frac{l^2}{4} - \frac{l^2}{2}\right]$$

$$= -\frac{Wl^2}{24EI}\left[-\frac{l^2}{4}\right]$$

$$= \frac{W.l^4}{96.EI}$$

$\therefore$ $\boxed{y_{max} = \frac{Wl^4}{96EI}}$ is the approximate Solution.

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