DEPARTMENT OF ELECTRONICS AND COMMUNICATION ENGINEERING

NETWORK ANALYSIS (J222E)

[R22]

B.TECH ECE
(II YEAR – I SEM)
(2023-24)

J. B. INSTITUTE OF ENGINEERING AND TECHNOLOGY
(UGC AUTONOMOUS)
Accredited by NBA & NAAC,
Approved by AICTE & Permanently affiliated to JNTUH
Bhaskar Nagar, Yenkapally(V), Moinabad(M), Ranaga Reddy(D),Hyderabad – 500 075**,**
Telanagana, India.

AY: 2022-23 Onwards	J. B. Institute of Engineering and Technology (UGC Autonomous)	B. Tech ECE II Year-I Sem			
Course Code: **L214C**	**SIGNALS AND SYSTEMS**	L	T	P	D
Credits: **3**		**3**	**0**	**0**	**0**

Pre-Requisites:
Mathematics, Basics of Probability

Module 1: Signal Analysis [10L]

Unit-I: Basic Signals [5L]
Basic elementary signals: Exponential and Sinusoidal signals, Ramp function, Gate function, Concepts of Impulse function, Unit Step function, Sinc Function, Signum function, Classification of signals.

Unit-II: Orthogonality Concept [5L]
Analogy between Vectors and Signals, Orthogonal Signal Space, Signal approximation using Orthogonal functions, Mean Square Error, Closed or complete set of orthogonal functions, Orthogonality in Complex functions.

Module 2: Fourier Analysis & Fourier Transform [12L]

Unit-I: Fourier analysis [5L]
Continuous time periodic signals, Fourier series representation. Trigonometric Fourier Series and Exponential Fourier Series, Properties of Fourier Series, Dirichlet's conditions, Complex Fourier spectrum.

Unit-II: Fourier Transform [7L]
Fourier Transform of arbitrary signal, standard signals, periodic signals, Fourier Transforms involving Impulse function and Signum function, Properties of Fourier Transform, Introduction to Hilbert Transform.

Module 3: Laplace Transforms [10L]

Unit-I: Direct Laplace Transform [6L]
Review of Laplace Transforms (LT), Concept of Region of Convergence (ROC), and Constraints on ROC for various classes of signals, Properties of LT

Unit-II: Inverse Laplace Transform [4L]
Inverse Laplace Transform, Partial fraction expansion method for inverse LT, Relation between LT and Fourier Transform.

Module 4: Signal Transmission through Linear Systems [10L]

Unit-I: Systems and its properties [5L]
System properties: linearity, additivity, homogeneity, shift-invariance, causality, stability, reliability. Linear Shift-Invariant (LSI) systems, characterization of causality and stability of LSI systems.
Transfer function of an LTI system.

Unit-II: Filter Characteristics [5L]
Filter characteristics of Linear Systems, Distortion less transmission through a system, Signal bandwidth, System bandwidth, and Causality and Paley-Wiener criterion for physical realization.

Module 5: Convolution and Correlation of Signals & Sampling [9L]

Unit-I: Convolution and Correlation [5L]
Concept of convolution in Time domain and Frequency domain, Graphical representation of Convolution, Cross Correlation and Auto Correlation of functions, Relation between Convolution and Correlation. Properties of Correlation function, Parseval's Theorem. Relation between Auto

Correlation function and Energy/Power spectral density function, Detection of periodic signals in the presence of Noise by Correlation, Extraction of signal from noise by filtering.

Unit-II: Sampling [4L]

Sampling theorem, graphical and analytical proof for Band Limited Signals, Impulse Sampling, Natural and Flat top Sampling, Reconstruction of signal from its samples, Effect of under sampling, aliasing, Introduction to Band Pass sampling.

Text Books

1. Signals, Systems & Communications - B.P. Lathi, 2009, BSP
2. Signals and Systems – A. Rama Krishna Rao – 2008, TMH.

Reference Books

1. Fundamentals of Signals and Systems - Michel J. Robert, 2008, MGH International Edition.
2. Signals and Systems - A.V. Oppenheim, A.S. Willsky and S.H. Nawab, 2nd ed.,PHI
3. Signals & Systems - Simon Haykin and Van Veen, Wiley, 2nd ed. Wiley.

E-Resources

10. https://nptel.ac.in/courses/108/106/108106163/
11. https://www.tutorialspoint.com/signals_and_systems/signals_analysis.htm

Course Outcomes

At the end of the course, the student will be able to:

CO1. Analyze the properties of Continuous signals and systems with their frequency response.
CO2. Develop input output relationship for linear shift invariant system and frequency response of continuous time using different Transforms.
CO3. Apply operations such as convolution and correlation for continuous time system.
CO4. Explain the concept of sampling in continues-time signals and apply sampling theorem in signal processing problems.
CO5. Discriminate the concepts of Fourier& Laplace Transforms as appropriate for various signals and systems.

CO-PO/PSO Mapping

Course Outcomes	Program Outcomes(POs)/Program Specific Outcomes(PSOs)													
	PO1	**PO2**	**PO3**	**PO4**	**PO5**	**PO6**	**PO7**	**PO8**	**PO9**	**PO10**	**PO11**	**PO12**	**PSO1**	**PSO2**
CO1	3	3	-	-	-	-	-	-	-	-	-	3	-	2
CO2	3	2	-	3	-	-	-	-	-	-	-	2	-	-
CO3	-	2	-	-	-	-	-	-	-	-	-	-	-	-
CO4	-	-	-	-	-	-	-	-	-	-	-	2	-	3
CO5	3	3	-	-	-	-	-	-	-	-	-	3	-	-
Average	3	2.5	-	3	-	-	-	-	-	-	-	2.5	-	2.5

Correlation: 3–Strong; 2–Medium; 1-Weak

UNIT - I

SIGNAL ANALYSIS AND FOURIER SERIES

Analogy between Vectors and Signals:

Before the concept, let us know the definition for signals and systems.

Signals: A function of one or more independent variables which contain some information.

(or)

Defined as any physical quantity that varies with time, space or any other independent variables.

Ex: Electric vltg or current. that includes radio sgl, TV sgl, telephone sgl etc.,
Non electric signals such as sound signal, pressure signal etc.,

→ A speech signal can be represented mathematically by acoustic (pertaining to the sense) pressure as function of time.

→ A picture can be represented by brightness as a function of two spatial variables.
(Relating to space)

Systems:

It is a set of elements or functional blocks that are connected together and produces an output in response to the input signal.

(or)

An entity that processes a set of input signals to yield another set of output signals.

Ex: An audio amplifier, attenuator, TV set, transmitter, receiver etc., or any engine or machine

Relation b/w signal & systems.

Ex: If we take example of central government, they we can see different sub-systems together considered as big system.

→ System is composed of many subsystems like finance, defence, foreign affairs, home culture, social welfare, industries etc.,

→ Inputs to the systems are in the form of revenue, import, complaints, business suggestions, policies for foreign countries through which central government functions.

→ The central government produces output signals in the form of exports, government resolutions, financial aids, welfare programs etc.,

→ This is the concept of signal and systems.

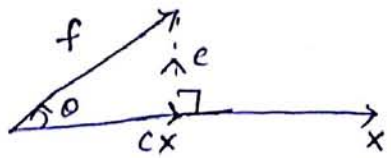
Analogy:

→ Signals are represented in terms of orthogonal functions.

Orthogonality Concept in Vectors:

→ All signals are basically vectors where vector is represented in terms of its co-ordinates.

→ Consider a vector f and another vector x then projection of vector along other vector is shown below



The dot product of vectors f and x is given as

$$f.x = |f||x|\cos\theta$$

where θ is the angle between f & x

cx is component of vector f along x or projection of f on x.

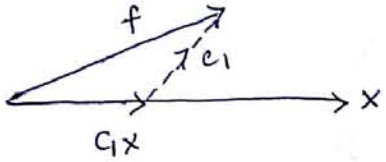
Using vector addition

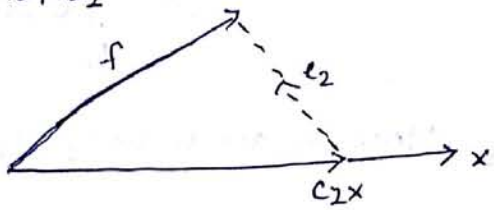
$$f = cx + e$$ → error vector

Note: 'e' is minimum only when it is perpendicular to x. Below are figures in which 'e' is not $\perp^{lar}$

$$f = c_1 x + e_1 = c_2 x + e_2$$

(Here e_1 & e_2 are > than e)





→ The component of f along x is cx which is given as $|f|\cos\theta$

$$\therefore c|x| = |f|\cos\theta$$

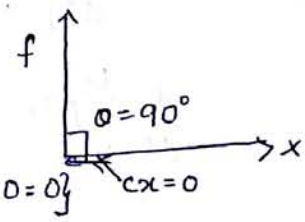
Multiplying both the sides by $|x|$

$$c|x|^2 = \underbrace{|f||x|\cos\theta}_{\text{dot product of vectors } f \text{ \& } x.}$$

$$c|x|^2 = f.x$$

$$c = \frac{f.x}{|x|^2} \Rightarrow \frac{f.x}{x.x}$$ {∵ x.x & f.x are vector product x and x cannot get cancelled}

→ When 'f' is $\perp^{lar}$ to x, 'f' will not have component along x because $\theta = 90°$ as shown in figure



$$f.x = |f||x|\cos\theta$$

$$= |f||x|\cos 90° \quad \{\because \cos 90 = 0\}$$

$$f.x = 0$$

→ The vectors 'f' and 'x' are said to be orthogonal if their dot product is zero. (or) vectors are orthogonal if they are mutually perpendicular.

Orthogonality in Signals:

Consider a signal f(t) to be represented in terms of x(t) over an interval t_1 & t_2.

$$f(t) = c\,x(t) + e(t)$$

$$e(t) = f(t) - c\,x(t) \qquad t_1 \leq t \leq t_2 \longrightarrow (1)$$

Energy of $e(t)$ will be

$$E_e = \int_{t_1}^{t_2} e^2(t)\,dt \rightarrow (2)$$

Mean square value of $e(t)$ will be

$$\overline{e^2(t)} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} e^2(t)\,dt \qquad \{\text{from eq (2)}\}$$

$$\overline{e^2(t)} = \frac{E_e}{t_2 - t_1}$$

From eq (1) we can write eqn (2)

$$E_e = \int_{t_1}^{t_2} [f(t) - c\,x(t)]^2\,dt$$

Here the value of 'c' should be selected such that E_e will be minimum

→ This can be obtained by differentiating E_e w.r.to c and equating it to zero

→ For minimum E_e, $\frac{dE_e}{dc} = 0$

$$\text{i.e} \quad \frac{d}{dc}\left[\int_{t_1}^{t_2} [f(t) - c\,x(t)]^2\right] dt = 0$$

$$\underbrace{\frac{d}{dc}\int_{t_1}^{t_2} f^2(t)\,dt}_{\substack{\text{Independent of 'c'} \\ \text{so it will be zero}}} - \frac{d}{dc}\int_{t_1}^{t_2} 2c\,f(t)\cdot x(t)\,dt + \frac{d}{dc}\int_{t_1}^{t_2} c^2 x^2(t)\,dt = 0$$

$$-2\int_{t_1}^{t_2} f(t)\cdot x(t)\,dt + 2c\int_{t_1}^{t_2} x^2(t)\,dt = 0$$

$$\cancel{2}\,c\int_{t_1}^{t_2} x^2(t)\,dt = \cancel{2}\int_{t_1}^{t_2} f(t)\cdot x(t)\,dt$$

$$c = \frac{\int_{t_1}^{t_2} f(t)\cdot x(t)\,dt}{\int_{t_1}^{t_2} x^2(t)\,dt}$$

The same expression can be obtained for minimum value of $\overline{e^2(t)}$.
The above equation denominator represents energy of $x(t)$. which cannot be zero. Hence numerator must be zero to make 'c' zero. If 'c' is zero, there will be no component of $f(t)$ along $x(t)$.

→ $f(t)$ and $x(t)$ are said to be orthogonal over an interval $[t_1, t_2]$ i.e

$$\int_{t_1}^{t_2} f(t)\, x(t)\, dt = 0$$

|| ly if $f(t)$ and $x(t)$ are complex signals then they are orthogonal over an interval $[t_1, t_2]$ if

$$\int_{t_1}^{t_2} f(t)\, \underbrace{x^*(t)}_{\text{complex conjugate of } x(t)}\, dt = 0 \quad \text{or} \quad \int_{t_1}^{t_2} \underbrace{f^*(t)}_{\text{complex conjugate of } f(x)}\, x(t)\, dt = 0$$

Problems:

(1) Show that the following signals are orthogonal over an interval $[0,1]$

$$f(t) = 1, \quad x(t) = \sqrt{3}(1-2t)$$

Sol we know for orthogonal if

$$\int_{t_1}^{t_2} x(t)\, f(t)\, dt = 0$$

$$\int_{t_1}^{t_2} f(t)\, x(t)\, dt = \int_0^1 1 \cdot (\sqrt{3})(1-2t)\, dt$$

$$= \int_0^1 \sqrt{3}\, dt - \int_0^1 2\sqrt{3}\, t\, dt$$

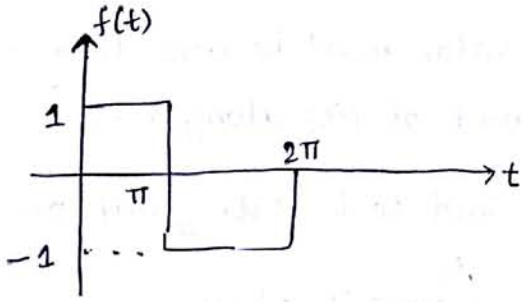
$$= \sqrt{3}\,[t]_0^1 - 2\sqrt{3}\left[\frac{t^2}{2}\right]_0^1$$

$$= \sqrt{3}\,[1-0] - 2\sqrt{3}\left[\frac{1}{2}\right]$$

$$= \sqrt{3} - \sqrt{3} = 0$$

Two given signals are orthogonal over an interval $[0,1]$

(2) Figure shows a square wave. Represent this signal by sint. plot an error in this representation.



Sol

Square wave be f(t). and sine wave be $x(t) = \sin t$. Then

$$f(t) = c \cdot x(t)$$
$$= c \cdot \sin t$$

value of c given by eqn

$$c = \frac{\int_{t_1}^{t_2} f(t)\, x(t)\, dt}{\int_{t_1}^{t_2} x^2(t)\, dt}$$

$$\int_{t_1}^{t_2} f(t)\, x(t)\, dt = \int_0^{2\pi} f(t) \cdot \sin t\, dt$$

$$= \int_0^{\pi} 1 \cdot \sin t\, dt + \int_{\pi}^{2\pi} (-1) \sin t\, dt$$

$$= [-\cos t]_0^{\pi} - [-\cos t]_{\pi}^{2\pi} = [-\{\cos \pi - \cos 0\}]$$

$$= [-(-1-1) + (1-(-1)] \quad + [\cos 2\pi - \cos \pi]$$

$$= 4.$$

$$\int_{t_1}^{t_2} x^2(t)\, dt = \int_0^{2\pi} \sin^2 t\, dt$$

$$= \int_0^{2\pi} \frac{1-\cos 2t}{2}\, dt = \frac{1}{2}\int_0^{2\pi} dt - \frac{1}{2}\int_0^{2\pi} \cos 2t\, dt$$

$$= \frac{1}{2}[t]_0^{2\pi} - \frac{1}{2}\left[\frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2} 2\pi - \frac{1}{4}[\sin 4\pi - \sin 0]$$

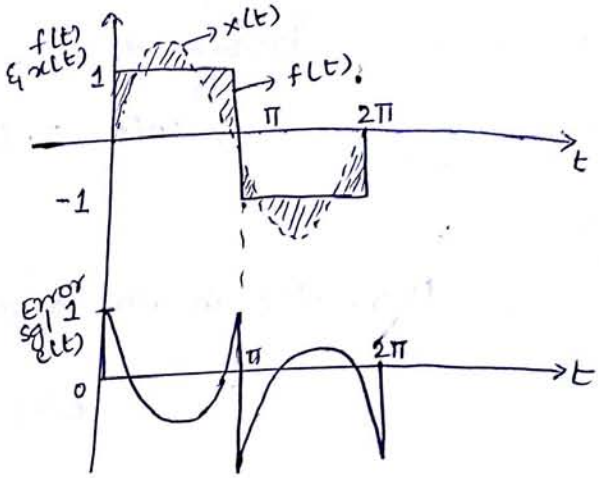
$$= \pi$$

$$\therefore \quad C = \frac{\int_{t_1}^{t_2} f(t)\, x(t)\, dt}{\int_{t_1}^{t_2} x^2(t)\, dt}$$

$$C = \frac{4}{\pi}$$

$$\therefore \quad f(t) = \frac{4}{\pi} \sin t$$

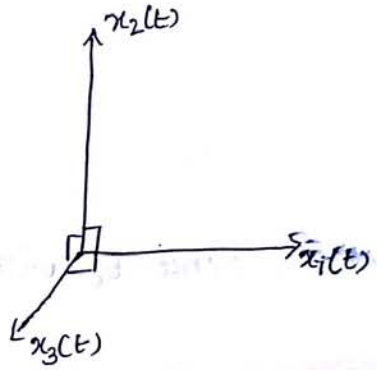
and error $e(t) = f(t) - c\,x(t)$



Orthogonal Signal Space:

Let $x_1(t)$, $x_2(t)$, $x_3(t)$ are orthogonal to each other i.e '3' signals are mutually $\perp^{lar}$ which forms three dimensional signal space which is also called as orthogonal signal space.

→ which is used to represent any signal lying in that space.



Note:

If there are 'N' such mutually orthogonal signals i.e $x_1(t), x_2(t), x_3(t), x_4(t) \ldots x_N(t)$ then they form N-dimensional orthogonal signal space.

Signal Approximation using Orthogonal functions:

→ Consider a set of signals which are mutually orthogonal over an interval $[t_1, t_2]$. $f(t)$ can be represented as

$$f(t) \cong c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t) + \cdots + c_N x_N(t)$$

$$\cong \sum_{n=1}^{N} c_n x_n(t)$$

In above eqn any two signals $x_m(t)$ and $x_n(t)$ are orthogonal over an interval $[t_1, t_2]$ i.e

$$\int_{t_1}^{t_2} x_m(t)\, x_n(t)\, dt = \begin{cases} 0 & \text{for } m \neq n \\ E_n & \text{for } m = n \end{cases}$$

because if $m=n$

$$\int_{t_1}^{t_2} x_n(t) \cdot x_n(t)\, dt = \int_{t_1}^{t_2} x_n^2(t)\, dt = E_n = \text{energy of the sgl.}$$

Error $e(t)$ in the approximation of equation is given as

$$e(t) = f(t) - \sum_{n=1}^{N} c_n x_n(t)$$

Hence error energy

$$E_e = \int_{t_1}^{t_2} e^2(t)\, dt = \int_{t_1}^{t_2} \left[f(t) - \sum_{n=1}^{N} c_n x_n(t) \right]^2 dt$$

where E_e is fn of $c_1, c_2 \cdots c_N$.

→ Hence E_e will be minimized w.r to c_i if

$$\frac{\partial E_e}{\partial c_i} = 0$$

$$\frac{\partial}{\partial c_i} \left\{ \int_{t_1}^{t_2} \left[f(t) - \sum_{n=1}^{N} c_n x_n(t) \right]^2 dt \right\} = 0$$

$$\frac{\partial}{\partial c_i} \left\{ \int_{t_1}^{t_2} f^2(t)\, dt - \int_{t_1}^{t_2} \sum_{n=1}^{N} 2 c_n f(t) x_n(t)\, dt + \int_{t_1}^{t_2} \sum_{n=1}^{N} c_n^2 x_n^2(t)\, dt \right] = 0$$

→ for $i = 1, 2, 3 \ldots N$ the equation is executed. The first integration term is independent of c_i so its derivative is zero.

So

$$\frac{\partial}{\partial c_i} \left[-\int_{t_1}^{t_2} 2 c_i f(t) x_i(t)\, dt + \int_{t_1}^{t_2} c_i^2 x_i^2(t)\, dt \right] = 0$$

$$\therefore -2 \int_{t_1}^{t_2} f(t) x_i(t)\, dt + 2 c_i \int_{t_1}^{t_2} x_i^2(t)\, dt = 0$$

$$c_i = \frac{\int_{t_1}^{t_2} f(t) x_i(t)\, dt}{\int_{t_1}^{t_2} x_i^2(t)\, dt} \quad \text{where } i = 1, 2, 3 \ldots N$$

we know that $\int_{t_1}^{t_2} x_i^2(t)\,dt = E_i$ = energy

$$\boxed{c_i = \frac{1}{E_i}\int_{t_1}^{t_2} f(t)\,x_i(t)\,dt}$$

Mean Square Error:

The error energy is given by equation

$$E_e = \int_{t_1}^{t_2}\left[f(t) - \sum_{n=1}^{N} c_n x_n(t)\right]^2 dt$$

$$E_e = \int_{t_1}^{t_2} f^2(t)\,dt - 2\int_{t_1}^{t_2}\sum_{n=1}^{N} c_n f(t)\,x_n(t)\,dt + \int_{t_1}^{t_2}\sum_{n=1}^{N} c_n^2 x_n^2(t)\,dt$$

Integration & Summation order if we interchange

$$E_e = \int_{t_1}^{t_2} f^2(t)\,dt - 2\sum_{n=1}^{N} c_n \int_{t_1}^{t_2} f(t)\,x_n(t)\,dt + \sum_{n=1}^{N} c_n^2 \int_{t_1}^{t_2} x_n^2(t)\,dt$$

$$E_e = \int_{t_1}^{t_2} f^2(t)\,dt - 2\sum_{n=1}^{N} c_n \cdot c_n E_n + \sum_{n=1}^{N} c_n^2 \cdot E_n \quad \left\{\because \int_{t_1}^{t_2} f(t)\,x_n(t)\,dt = c_n E_n \;\&\; \int_{t_1}^{t_2} x_n^2(t) = E_n\right\}$$

$$= \int_{t_1}^{t_2} f^2(t)\,dt - 2\sum_{n=1}^{N} c_n^2 \cdot E_n + \sum_{n=1}^{N} c_n^2 \cdot E_n$$

$$= \int_{t_1}^{t_2} f^2(t)\,dt - \sum_{n=1}^{N} c_n^2 \cdot E_n.$$

The mean square error and error energy are related as

$$\overline{e^2(t)} = \frac{E_e}{t_2 - t_1} \Rightarrow \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} f^2(t)\,dt - \sum_{n=1}^{N} c_n^2 \cdot E_n\right]$$

$\therefore \sum_{n=1}^{N} c_n^2 \cdot E_n$ is always positive so if $E_e \to 0$ as $N \to \infty$

Closed or Complete set of Orthogonal Functions:

→ Mean square error approaches to zero as number of terms $C_n^r E_n$ are made infinite. Under this condition,

with $\overline{e^r(t)} = 0$ as $N = \infty$

$$0 = \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} f^r(t)\,dt - \sum_{n=1}^{N} C_n^r \cdot E_n\right]$$

$$\therefore \int_{t_1}^{t_2} f^r(t)\,dt = \sum_{n=1}^{\infty} C_n^r \cdot E_n.$$

from

$$f(t) = \sum_{n=1}^{N} C_n x_n(t) \quad \text{when } N \to \infty$$

$$\therefore f(t) = \sum_{n=1}^{\infty} C_n x_n(t). \quad \{\text{Generalized fourier Series}\}$$

→ It is said to be complete or closed set if there exists no function $p(t)$ for which

$$\int_{t_1}^{t_2} p(t)\, x_n(t)\, dt = 0 \quad \text{for } n = 1, 2 \ldots$$

→ If $p(t)$ exists and above integral is zero, then $p(t)$ must be member of set $\{x_n(t)\}$

→ For complete set, function $f(t)$ is expressed as

$$f(t) = C_1 x_1(t) + C_2 x_2(t) + C_3 x_3(t) + \ldots$$

$$C_i = \frac{\int_{t_1}^{t_2} f(t)\, x_i(t)\, dt}{\int_{t_1}^{t_2} x_i^r(t)\, dt} = \frac{1}{E_i}\int_{t_1}^{t_2} f(t)\, x_i(t)$$

Orthogonality in Complex functions:

Let set of signals $x_1(t), x_2(t), x_3(t) \ldots$ are complex. Then those signals are mutually orthogonal if

$$\int_{t_1}^{t_2} x_m(t)\, x_n^*(t)\, dt = \int_{t_1}^{t_2} x_m^*(t)\, x_n(t)\, dt = \begin{cases} 0 & \text{for } m \neq n \\ E_n & \text{for } m = n. \end{cases}$$

Then $f(t)$ can be expressed as

$$f(t) = \sum_{n=1}^{\infty} C_n x_n(t)$$

$$\text{where } C_n = \frac{1}{E_n} \int_{t_1}^{t_2} f(t)\, x_n^*(t)\, dt$$

$$E_n = \int_{t_1}^{t_2} x_n(t) \cdot x_n^*(t)\, dt.$$

Problems:

(1) Show that the signal set $\{1, \cos\omega_0 t, \cos 2\omega_0 t, \ldots \cos n\omega_0 t, \ldots \sin\omega_0 t, \sin 2\omega_0 t \ldots \sin n\omega_0 t\}$ are orthogonal over an interval $T_0 = \frac{2\pi}{\omega_0}$.

Sol (i) To check orthogonality of cosine waves:

Consider the orthogonality of $\cos n\omega_0 t$ and $\cos m\omega_0 t$ i.e

$$\int_{t}^{t+T_0} \cos n\omega_0 t \cos m\omega_0 t\, dt$$

$$\left\{ \because \cos x \cos y = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right] \right.$$

$$\int_{t}^{t+T_0} \cos n\omega_0 t \cos m\omega_0 t\, dt = \frac{1}{2}\int_{t}^{t+T_0} \cos(n-m)\omega_0 t\, dt + \frac{1}{2}\int_{t}^{t+T_0} \cos(n+m)\omega_0 t\, dt.$$

For $n=m$, $\cos(n-m)\omega_0 t = 1$ but for $n \neq m$, the integration of $(n-m)$ full cycles of cosine wave is taken over one period. Hence integration is zero. Similarly for integration of $(n+m)$ full cycles

$\therefore$ Equation becomes.

$$\int_t^{t+T_0} \cos n\omega_0 t \cdot \cos m\omega_0 t \, dt = \frac{1}{2}\int_t^{t+T_0} 1 \, dt$$

$$= \frac{1}{2}\left[t\right]_t^{t+T_0} = \frac{T_0}{2}.$$

$$\therefore \int_t^{t+T_0} \cos n\omega_0 t \cos m\omega_0 t \, dt = \begin{cases} 0 & \text{for } n \neq m \\ \frac{T_0}{2} & \text{for } n = m. \end{cases}$$ where it shows that

two cosine waves of given set are orthogonal over one period.

(ii) To check orthogonality of sine waves.

$$\int_t^{t+T_0} \sin n\omega_0 t \sin m\omega_0 t \, dt = \frac{1}{2}\int_t^{t+T_0} \cos(n-m)\omega_0 t \, dt + \frac{1}{2}\int_t^{t+T_0} \cos(n+m)\omega_0 t \, dt$$

$$\left\{\because \sin x \sin y = \frac{1}{2}\{\cos(x-y) - \cos(x+y)\}\right\}$$

Same as above explanation

$$\therefore \int_t^{t+T_0} \sin n\omega_0 t \sin m\omega_0 t \, dt = \frac{1}{2}\int_t^{t+T_0} 1 \, dt = \frac{T_0}{2}.$$

$$\int_t^{t+T_0} \sin n\omega_0 t \sin m\omega_0 t \, dt = \begin{cases} 0 & \text{for } n \neq m \\ \frac{T_0}{2} & \text{for } n = m. \end{cases}$$

(iii) To check orthogonality of $\sin n\omega_0 t$ and $\cos m\omega_0 t$

$$\int_t^{t+T_0} \sin n\omega_0 t \cos m\omega_0 t \, dt = \frac{1}{2}\int_t^{t+T_0} \sin(n-m)\omega_0 t \, dt + \frac{1}{2}\int_t^{t+T_0} \sin(n+m)\omega_0 t \, dt$$

$$\left\{\because \sin x \cos y = \frac{1}{2}\left[\sin(x-y) + \sin(x+y)\right]\right\}$$

Integration of $(n-m)$ or $(n+m)$ full cycles of sine wave over a period will be zero. hence both the above integrals are zero.

$$\therefore \int_t^{t+T_0} \sin n\omega_0 t \cos n\omega_0 t \, dt = 0$$ for all values of n & m.

$\therefore$ Thus sine & cosine waves of given set are orthogonal over one period.

(2) Prove that set of exponentials $1, e^{\pm j\omega_0 t}, e^{\pm 2j\omega_0 t}, e^{\pm j3\omega_0 t} \ldots$ is orthogonal over any interval T_0.

Sol Here we have to check orthogonality of complex function. It is given as

$$\int_{t_1}^{t_2} x_m(t)\, x_n^*(t)\, dt = \begin{cases} 0 & \text{for } m \neq n \\ E_n & \text{for } m = n \end{cases}$$

For $x_m(t) = e^{jm\omega_0 t}$, $x_n(t) = e^{jn\omega_0 t}$.

$$\int_t^{t+T_0} e^{jm\omega_0 t} \left[e^{jn\omega_0 t}\right]^* dt = \int_t^{t+T_0} e^{jm\omega_0 t} \cdot e^{-jn\omega_0 t}\, dt$$

$$= \int_t^{t+T_0} e^{j(m-n)\omega_0 t}\, dt \qquad \left(\because \int e^x \Rightarrow \frac{1}{x} \cdot e^x\right)$$

$$= \frac{1}{j(m-n)\omega_0}\left[e^{j(m-n)\omega_0(t+T_0)} - e^{+j(m-n)\omega_0 t}\right]$$

$$= \frac{1}{j(m-n)\omega_0}\left[e^{j(m-n)\omega_0 t} \cdot e^{j(m-n)\omega_0 T_0} - e^{j(m-n)\omega_0 t}\right]$$

$$= \frac{1}{j(m-n)\omega_0} e^{j(m-n)\omega_0 t}\left[e^{j(m-n)\omega_0 T_0} - 1\right]$$

Here $\omega_0 = \frac{2\pi}{T_0}$ $\therefore \omega_0 T_0 = 2\pi$

$$\int_t^{t+T_0} e^{jm\omega_0 t}\left[e^{jn\omega_0 t}\right]^* dt = \frac{1}{j(m-n)\omega_0} e^{j(m-n)\omega_0 t}\left[1 - 1\right]$$

$$= 0 \qquad \left\{\because e^{j(m-n)\cdot 2\pi} = 1 \text{ always}\right\}$$

Thus complex exponentials are orthogonal over any time period T_0.

Now when $n = m$ i.e

$$\int_t^{t+T_0} e^{jm\omega_0 t}\left[e^{jn\omega_0 t}\right]^* dt = \int_t^{t+T_0} e^{j(m-m)\omega_0 t}\, dt = \int_t^{t+T_0} 1\, dt$$

$$= [t]_t^{t+T_0} = T_0$$

$$\int_{t}^{t+T_0} e^{jm\omega_0 t} e^{-jn\omega_0 t}\, dt = \begin{cases} 0 & \text{for } m \neq n \\ T_0 & \text{for } m = n \end{cases}$$

$\downarrow$

Energy of complex exponential fn.

(3) If $x(t)$ and $y(t)$ are orthogonal then show that the energy of the signal $x(t)+y(t)$ is identical to the energy of the signal $x(t)$ plus energy of the signal $y(t)$.

Sol Let energy of $x(t)$ be E_x & energy of $y(t)$ be E_y i.e

$$E_x = \int_{-\infty}^{\infty} x^2(t)\, dt \quad ; \quad E_y = \int_{-\infty}^{\infty} y^2(t)\, dt.$$

Energy of sum of signal i.e $x(t)$ and $y(t)$ will be

$$\int_{-\infty}^{\infty} [x(t)+y(t)]^2\, dt = \int_{-\infty}^{\infty} [x^2(t)+y^2(t)+2x(t)y(t)]\, dt$$

$$= \int_{-\infty}^{\infty} x^2(t)\, dt + \int_{-\infty}^{\infty} y^2(t)\, dt + 2\int_{-\infty}^{\infty} x(t)\, y(t)\, dt$$

Since $x(t)$ & $y(t)$ are orthogonal, third integration term in above eqn will be zero

$$\int_{-\infty}^{\infty} [x(t)+y(t)]^2\, dt = \int_{-\infty}^{\infty} x^2(t)\, dt + \int_{-\infty}^{\infty} y^2(t)\, dt$$

$$= E_x + E_y$$

$\therefore$ Sum of energies of orthogonal is equal to energy of the total sum of signals.

(4) Show that over the period of interval '0' to 2π, a rectangular function is orthogonal to signals $\cos t, \cos 2t \ldots \cos nt$ for all integers values of n.

Sol Rectangular function over the period 0 to 2π

$$f(t) = \begin{cases} 1 & \text{for } 0 \le t \le 2\pi \\ 0 & \text{for others} \end{cases}$$

$$x(t) = \cos nt$$

$$\int_t^{t+T_0} f(t)\, x(t)\, dt = \int_t^{t+T_0} 1 \cdot \cos nt\, dt$$

$$= \int_t^{2\pi} \cos nt\, dt \Rightarrow \left[\frac{\sin nt}{n}\right]_0^{2\pi} \Rightarrow \frac{1}{n}\left[\sin 2\pi t - \sin 0\right] = 0$$

$\therefore$ $\cos nt$ and rectangular fn are orthogonal over an interval 0 to 2π

(5) Show that the sequence $e^{j2\pi kn/N}$ is an orthogonal sequence, periodic in N.

Sol Given sequence $x(n) = e^{j2\pi kn/N}$

It will be periodic if $x(n+N) = x(n)$

$$x(n+N) = e^{j2\pi k(n+N)/N} = e^{j2\pi kn/N} \cdot e^{j2\pi k}$$

Here $e^{j2\pi k} = \cos 2\pi k + j\sin 2\pi k$

$= 1 + 0 = 1$ always

$$x(n+N) = e^{j2\pi kn/N} = x(n)$$

$\therefore$ $x(n)$ is periodic with period N.

Let two sequences be $x_k(n) = e^{j2\pi kn/N}$ and $x_l(n) = e^{j2\pi ln/N}$

Orthogonality of discrete time sequences can be checked over one period

$$\sum_{n=0}^{N-1} x_k(n)\, x_l^*(n) = \sum_{n=0}^{N-1} e^{j2\pi kn/N} \left[e^{j2\pi ln/N}\right]^*$$

$$= \sum_{n=0}^{N-1} e^{j2\pi kn/N} \cdot e^{-j2\pi ln/N} = \sum_{n=0}^{N-1} e^{j2\pi\left(\frac{k-l}{N}\right)n}.$$

Standard series formula

$$\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1-a} \quad \{N_2 > N_1\}$$

Here $a = e^{j2\pi\left(\frac{k-l}{N}\right)}$

$$\sum_{n=0}^{N-1} x_k(n)\, x_l^*(n) = \frac{\left[e^{j2\pi\frac{k-l}{N}}\right]^0 - \left[e^{j2\pi\left(\frac{k-l}{N}\right)}\right]^N}{1-e^{j2\pi\frac{k-l}{N}}} = \frac{1-e^{j2\pi(k-l)}}{1-e^{j2\pi\frac{k-l}{N}}}.$$

k & l are integers, so $k-l$ will also be an integer. Therefore $e^{j2\pi(k-l)} = 1$ always

$$\sum_{n=0}^{N-1} x_k(n)\, x_l^*(n) = \frac{1-1}{1-e^{j2\pi\frac{k-l}{N}}} = 0$$

(6) Find if following signals are orthogonal $x_1(n) = e^{jk\left(\frac{\pi}{8}\right)n}$ and $x_2(n) = e^{jm\left(\frac{\pi}{8}\right)n}$

Sol) $x_1(n) = e^{jk\left(\frac{\pi}{8}\right)n}$ & $x_2(n) = e^{jm\left(\frac{\pi}{8}\right)n}$

$$\sum_{n=0}^{N-1} x_1(n)\, x_2^*(n) = \sum_{n=0}^{N-1} e^{j2\pi kn/16} \cdot e^{-j2\pi mn/16} = \sum_{n=0}^{N-1} e^{j2\pi(k-m)n/16}$$

$$= \frac{\left[e^{j2\pi(k-m)/16}\right]^0 - \left[e^{j2\pi(k-m)/16}\right]^N}{1-e^{j2\pi(k-m)/16}}$$

Here $N = 16$

$$= \frac{1-e^{j2\pi(k-m)}}{1-e^{j2\pi(k-m)/16}}.$$

but $e^{j2\pi(k-m)} = 1$ always so

$$\sum_{n=0}^{N-1} x_1(n)\, x_2^*(n) = 0$$

$\therefore$ two sgls are orthogonal.

Classification of Signals:

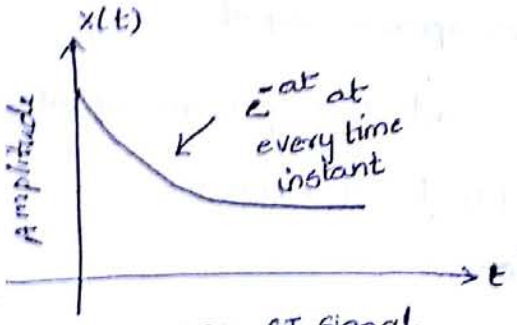
→ Signals classified into two types depending on independent variable time

a) Continuous Time (CT) signals

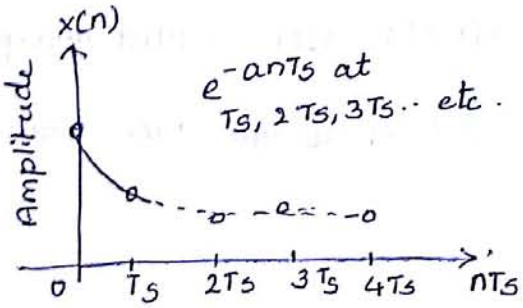
b) Discrete Time (DT) signals.

CT signals & DT signals:

It is defined continuously with respect to time. A DT signal is defined only at specific or regular time instants.



(a) CT signal



(b) DT signal.

Significance:

(i) Analog circuits process CT signals. Such circuits are op-amps, filters, amplifier etc.,

(ii) Digital circuits process DT signals. Such circuits are microprocessors, counters, flipflops etc.,

→ When amplitude of CT signal varies continuously it is called analog sgl. In other words amplitude and time are continuous for analog signal.

→ When amplitude of DT signal takes only finite varies it is digital signal.

Periodic and Non-Periodic Signals:

→ A signal is said to be periodic if it repeats at regular intervals.

(or)

→ A signal which repeats after every time interval T is called periodic signal.

$x(t)$ is called periodic if and only if

$$x(t+T) = x(t) \text{ for all } t.$$

time constant

→ The smallest value of T that satisfies this condition is called fundamental period or simple period of $x(t)$.

→ The reciporal of fundamental period T is called fundamental frequency f of $x(t)$

$$f = \frac{1}{T}$$

$$\text{Angular frequency} = \omega = 2\pi f = \frac{2\pi}{T}$$

$$T = \frac{2\pi}{\omega}$$

→ A signal $x(t)$ for which there is no value of T satisfying the condition $x(t+T) = x(t)$ is called non-periodic or aperiodic signal.

|| ly for discrete-time signals, $x[n]$ is said to be periodic if it satisfies

$$x[n+N] = x[n] \text{ for all integers 'n'}$$

N ↓ +ve integer

$$\text{Angular frequency } \Omega = \frac{2\pi}{N}$$

$$N = \frac{2\pi}{\Omega}$$

Note :

(1) Sum of two continuous-time periodic signals may not be periodic

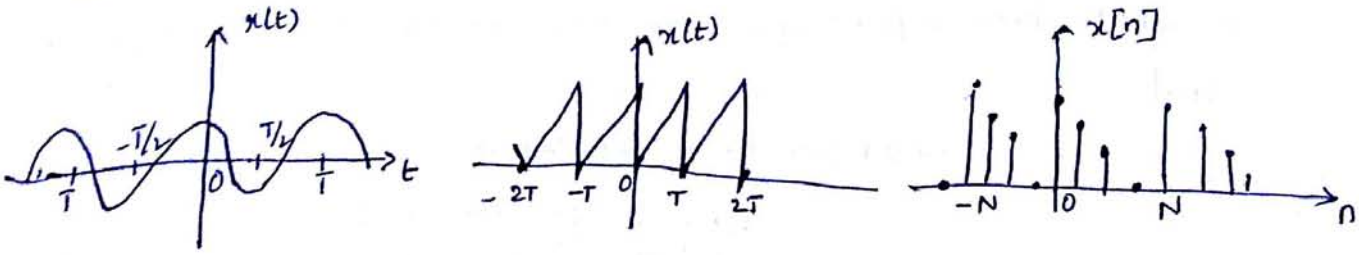
(2) Sum of two periodic sequences is always periodic.

(3) Sum of two periodic signals is periodic only if the ratio of their respective periods is a rational number.

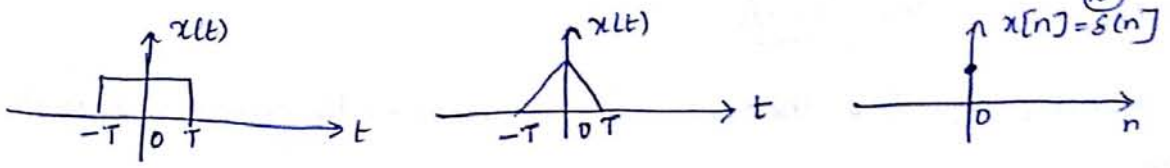
$$\frac{T_1}{T_2} = \text{rational number.}$$

(4) Fundamental period is the LCM of T_1 & T_2.

(5) If the ratio T_1/T_2 is an irrational number, then signals $x_1(t)$ & $x_2(t)$ do not have a common period and $x(t)$ cannot be periodic.



(a) periodic



(b) Non-periodic

Even and Odd Signals:

→ A signal $x(t)$ or $x[n]$ is said to be an even sgl if it satisfies the condition

$$x(-t) = x(t) \text{ for all } t,$$

$$x[-n] = x[n] \text{ for all } n.$$

→ A signal $x(t)$ or $x[n]$ is said to be an odd sgl if it satisfies the condition

$$x(-t) = -x(t) \text{ for all 't'}$$

$$x[-n] = -x[n] \text{ for all 'n'.}$$

→ Even signals are symmetrical abt vertical axis or time origin whereas odd signals are asymmetric.

cosine

sine

→ A signal $x(t)$ or $x[n]$ can be expressed as sum of two signals ie one odd & one even.

$$x(t) = x_e(t) + x_o(t)$$

$$x[n] = x_e[n] + x_o[n]$$

where

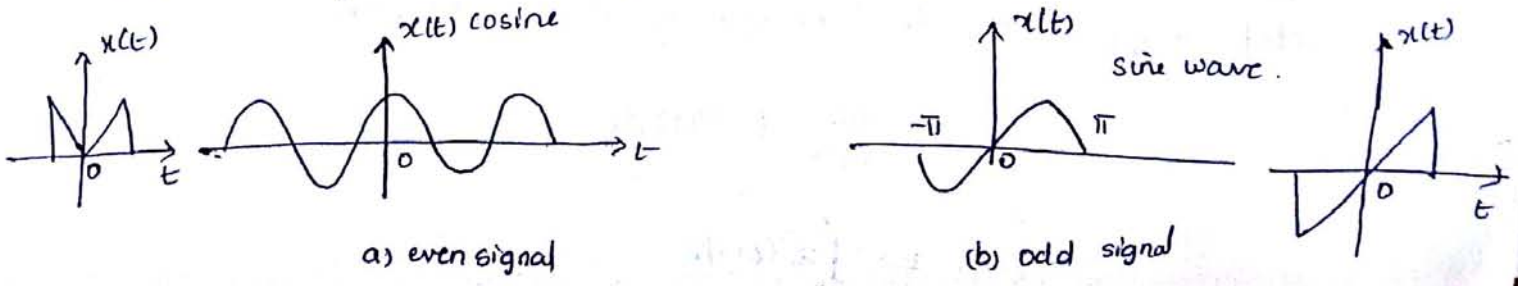
$$x_e[n] = \frac{1}{2}\{x(n) + x(-n)\}, \text{ even part}$$

$$x_o[n] = \frac{1}{2}[x(n) - x(-n)]$$

$$x_e(t) = \frac{1}{2}\{x(t) + x(-t)\}$$

$$x_o(t) = \frac{1}{2}\{x(t) - x(-t)\}$$

→ The product of two even or odd signals is an even signal & product of an even signal and odd signal is an odd signal.



a) even signal

(b) odd signal

Deterministic and Random Signals

→ A deterministic signal is the one where no uncertainity occurs w.r. to its value at any time.

$$x(t) = 100 \sin 50t \text{ (Continuous)}$$

$$x[n] = 100 \sin 50n \text{ (Discrete)}$$

→ A random signal is the one about which there is some degree of uncertainity before it actually occurs.

For example : the o/p of TV/radio receiver when tuned to frequency where there is no broadcast.

Real And Complex Signals:

→ $x(t)$ is real signal if its value is a real number and is a complex signal if its value is a complex number.

Eg: $x(t) = x_1(t) + j x_2(t)$

real signals and $j = \sqrt{-1}$

Energy and Power Signals:

→ In electrical systems, signals may represent current or vltg.

Consider a voltage signal $v(t)$ across resistor 'R' producing current $i(t)$

Then power dissipated in resistor is given by

$$P(t) = \frac{V^2(t)}{R} = i^2(t) \cdot R.$$

when $R = 1\Omega$

$$P(t) = V^2(t) = i^2(t).$$

In general $x(t)$ whether it is vltg or current sgl we get power given by

$$P(t) = x^2(t).$$

Total energy or normalized energy E of sgl $x(t)$ is defined by

$$E = \lim_{T \to \infty} \int_{-T/2}^{T/2} x^2(t)\, dt.$$

$$\therefore E = \int_{-\infty}^{\infty} x^2(t)\, dt$$

The average power or normalized average power P of the signal $x(t)$ is

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\,dt$$

→ In case of discrete-time signal $x[n]$, integrals replaced by summation

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Note:

→ The signals for which total energy is finite $(0<E<\infty)$ are called energy signals. They have zero average power.

Ex: deterministic & non-periodic sgls.

→ The signals for which the average power is finite $(0<P<\infty)$ are called powersignals. They have infinite energy Ex: Random, periodic sgl.

→ Both energy & power signals are mutually exclusive.

Elementary Signals:

Unit step function:

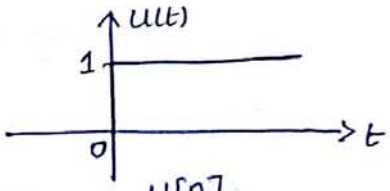
↓

Important signal used in many cases. Ex: When we apply brake to an automobile we are applying constant force.

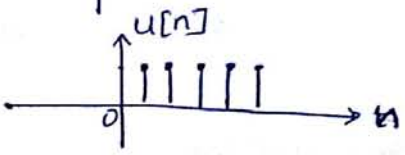
→ If a step function has unity magnitude then it is called unit step fn.

It is defined as

$$u(t) = 1 \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t<0$$

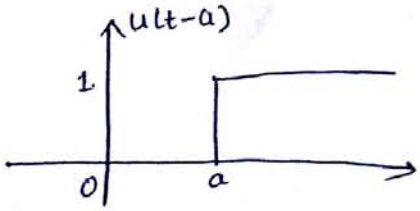


discrete $u[n] = 1$ for $n \geq 0$
0 for $n<0$.



$||^{ly}$ for shifted unit step fn $u(t-a)$ is zero if $t-a<0$ or $t<a$. and $t-a>0$ or $t>a$.

$$\therefore \quad u(t-a) = 1 \quad \text{for } t>a$$
$$= 0 \quad \text{for } t<a$$



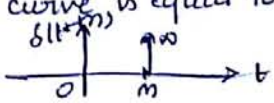
Impulse function (Dirac Delta fn) :

It is defined as

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \text{ at } t=0 \text{ and } \delta(t) = 0 \text{ for } t \neq 0$$

i.e The function has zero amplitude everywhere except at $t=0$. At $t=0$ amplitude is infinity such that area under the curve is equal to one.

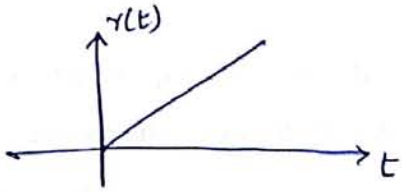
$$\delta(t-m) = \infty \; ; \; t = m \quad \& \quad \int_{-\infty}^{\infty} \delta(t-m)\, dt = 1$$
$$= 0 \quad t \neq m$$



Unit Ramp Function :

Unit ramp is defined as

$$r(t) = t \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0$$



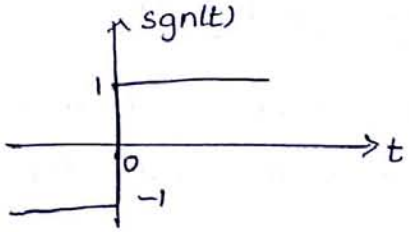
→ Ramp fn can be obtained by applying unit step fn to integrator

$$r(t) = \int u(t)\, dt = \int dt = t \quad (\text{in interval } t>0).$$

Signum function :

It is defined by

$$\text{sgn}(t) = \begin{cases} 1 & , t > 0 \\ -1 & , t < 0 \\ 0 & , t = 0 \end{cases}$$



→ This fu in terms of unit step fn

$$\text{sgn}(t) = 2u(t) - 1$$

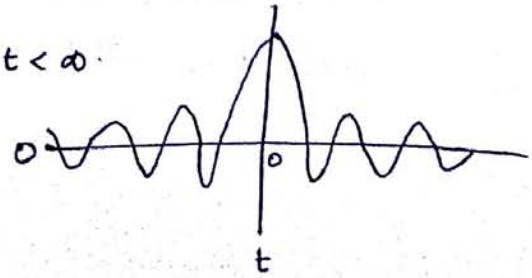
For $t>0$ $\quad 2u(t) = 2$

for $t<0$ $\quad 2u(t) = 0$.

Sinc function :

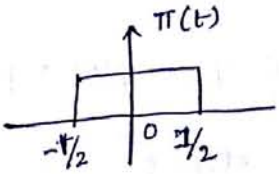
The sinc fn defined by expression

$$\text{sinc}(t) = \frac{\sin t}{t} \qquad -\infty < t < \infty.$$



Rectangular pulse function:

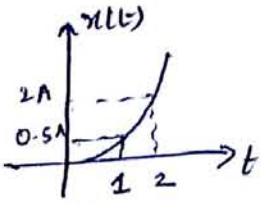
$$\Pi(t) = 1 \text{ for } |t| \leq \frac{1}{2}$$
$$= 0 \text{ otherwise}$$



Parabolic signal:

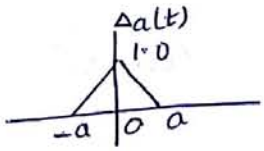
defined as

$$x(t) = \frac{At^2}{2} \text{ for } t \geq 0$$
$$= 0 \; ; t < 0$$



Triangular pulse function:

$$\Delta_a(t) = \begin{cases} 1 - \frac{|t|}{a} & |t| \leq a \\ 0 & |t| > a \end{cases}$$

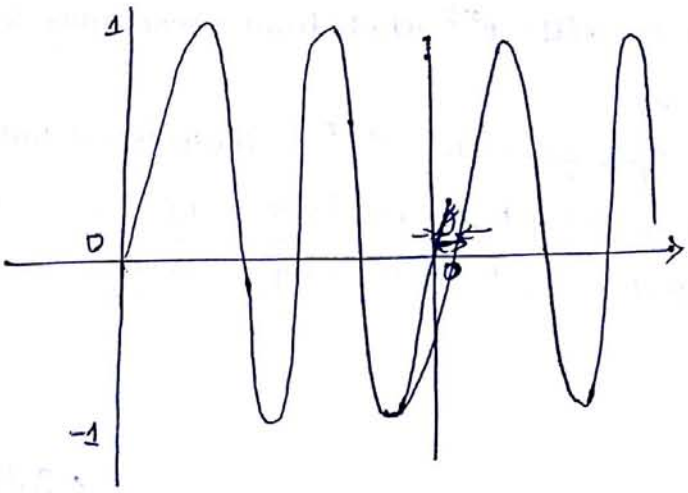


Sinusoidal signal: / cosinusoidal signal $x(t) = A\cos(\Omega t + \phi)$

A continuous-time sinusoidal signal is given by

$$x(t) = A\sin(\Omega t + \theta)$$

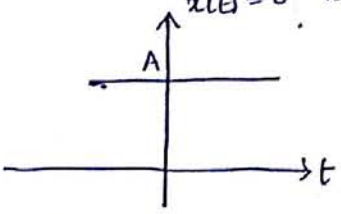
A → amplitude; Ω → frequency in radians per second; θ → phase angle in radians.



$A = 1$, $\theta = -\frac{\pi}{3}$, time period of $\frac{2\pi}{\Omega}$.

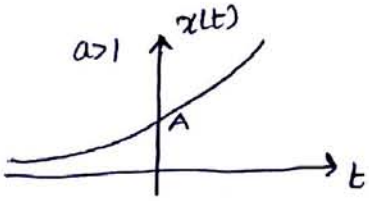
Real Exponential signals:

$x(t) = e^{at}$ for $a = 0$



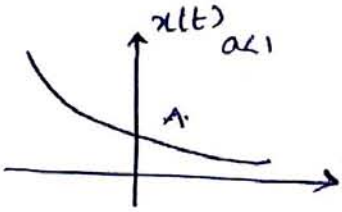
a) dc signal

'a' is positive



b) exponentially growing

'a' is negative



c) exponentially decaying

A real exponential signal is defined as

$$x(t) = Ae^{at}$$

where A, a are real.

Complex exponential signal:

The most general form of complex exponential is given by

$$x(t) = e^{st}$$

where s = complex variable $= \sigma + j\Omega = s$

$$x(t) = e^{st} = e^{(\sigma + j\Omega)t} = e^{\sigma t} \cdot e^{j\Omega t} \rightarrow (1)$$

Using Euler's identity

$$e^{j\Omega t} = \cos\Omega t + j\sin\Omega t \rightarrow (2)$$

Substitute (2) in (1)

$$x(t) = e^{\sigma t}[\cos\Omega t + j\sin\Omega t]$$

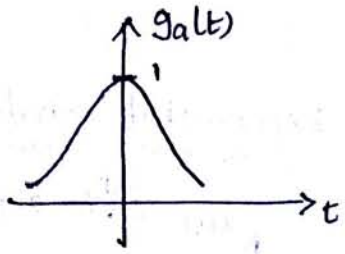
→ Depending on the values of σ and Ω we get different signals.

1. If $\sigma = 0$, $\Omega = 0$; $x(t) = 1$; pure DC signal
2. If $\Omega = 0$ then $s = \sigma$, $x(t) = e^{\sigma t}$ which decay exponentially for $\sigma < 0$ & grows exponentialy for $\sigma > 0$.
3. If $\sigma = 0$ then $s = \pm j\Omega$ gives $x(t) = e^{j\Omega t}$ a sinusoidal sgl with $\phi = 0$.
4. If $\sigma < 0$ then finite Ω we get exponentially decaying sinusoidal signal.
5. If $\sigma > 0$ with finite Ω, we get exponentially growing sinusoidal signal.

Gaussian Signal

It is defined as

$$x(t) = g_a(t) = e^{-a t^2} ; \quad -\infty < t < \infty$$



UNIT - II

FOURIER TRANSFORMS AND SAMPLING

FOURIER TRANSFORMS:

→ In Fourier series any continuous time periodic signal f(t) can be represented as linear combination of complex exponentials and fourier coefficients are discrete.

→ The F.S can be applied to periodic signals only but F.T can also be applied to non periodic functions like rectangular pulse, step fn, ramp fn etc.

→ The fourier transform can be developed by finding fourier series of periodic function and then tending T to infinity.

→ If T tends to infinity then f(t) becomes aperiodic signal.

DERIVING FOURIER TRANSFORM FROM FOURIER SERIES:

Exponential form of fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t}\, dt$$

for aperiodic

$$TF_n = \int_{-\infty}^{\infty} f(t) e^{-jn\omega_0 t}\, dt \rightarrow (1)$$

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt} \longrightarrow F.T \rightarrow (2)$$

The coefficient F_n becomes by comparing (1) & (2)

$$TF_n = F(\omega)$$

$$F_n = \frac{1}{T} F(\omega)$$

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} \cdot F(\omega)\, e^{jn\omega_0 t}$$

Other wise

$$x(t) = \sum_{k=-\infty}^{\infty} x(n) e^{jn\omega t} \rightarrow (1)$$

$$x(n) = \frac{1}{T}\int_{\langle T\rangle} x(t) e^{-jn\omega t}\, dt$$

$$Tx(n) = \int_{\langle T\rangle} x(t) e^{-jn\omega t}\, dt \rightarrow (2)$$

$\omega = n\omega_0$ → continuous → discrete

Reagranging eq (1)

$$x(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} Tx(n) e^{jn\omega t}$$

we know $\omega = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\omega}$

$$x(t) = \frac{\omega}{2\pi}\sum_{k=-\infty}^{\infty} Tx(n) e^{jn\omega t}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} Tx(n) e^{jn\omega t} \cdot \omega$$

As $T \to \infty$, ω becomes small. Hence $\omega \to d\omega$

→ $Tx(k)$ becomes continuous as $T \to \infty$

$$x(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

Eq ② can be written as

$T \to \infty$, $T\,X(\omega) \to X(\omega)$

eq ②

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$f(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} F(n\omega_0)\right] e^{jn\omega_0 t}$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} F(n\omega_0)\, e^{jn\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T} \quad ; \quad T = \frac{2\pi}{\omega_0}$$

$$f(t) = \frac{\omega_0}{2\pi} \sum_{n=-\infty}^{\infty} F(n\omega_0)\, e^{jn\omega_0 t}.$$

$\left(\because \omega_0 = \frac{\omega}{n} \quad n\,d\omega_0 = d\omega. \right)$

$n \to \infty$

$n\omega_0 \to$ approaches a continuous freq variable 'ω'.

$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega}$$

IFT.

Note :

→ For periodic signals discrete waveform is formed and for aperiodic signals continuous waveform is formed.

→ Fourier transform of a signal $f(t)$ is defined as

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

Inverse fourier transform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega.$$

$$F(\omega) = \underbrace{|F(\omega)|}_{\text{magnitude}}\, e^{j\underbrace{\theta(\omega)}_{\text{phase}}}$$

→ By application of F.T a signal in time domain is converted into frequency domain.

FOURIER TRANSFORM OF ANY ARBITARY SIGNALS AND STANDARD SIGNALS:

(1) Fourier transform of Single Sided Exponential signal.

The single sided exponential is defined as

$$x(t) = Ae^{-at} \quad ; \text{ for } t \geq 0$$

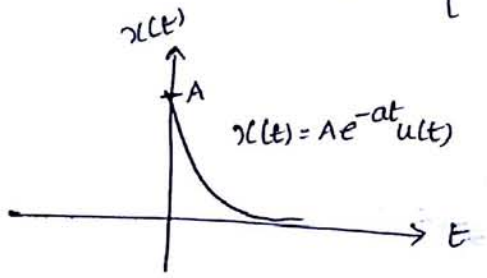
By definition of fourier transform

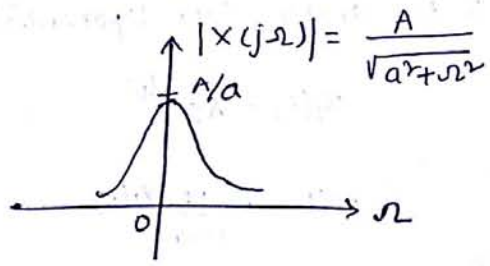
$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t} dt = \int_{0}^{\infty} A e^{-at} e^{-j\Omega t} dt$$

$$= \int_{0}^{\infty} A e^{-(a+j\Omega)t} dt = \left[\frac{A e^{-(a+j\Omega)t}}{-(a+j\Omega)} \right]_0^{\infty}$$

$$= \left[\frac{A e^{-\infty}}{-(a+j\Omega)} - \frac{A e^{0}}{-(a+j\Omega)} \right] = \frac{A}{a+j\Omega}.$$

$$\therefore F\{A e^{-at} u(t)\} = \frac{A}{a+j\Omega}$$





(2) Fourier transform of Doubled Sided Exponential Signal

$$x(t) = A e^{-a|t|} \quad ; \text{ for all 't'}$$

$$\therefore x(t) = Ae^{at} \quad ; \text{ for } t = -\infty \text{ to } 0$$

$$= Ae^{-at} \quad ; \text{ for } t = 0 \text{ to } \infty$$

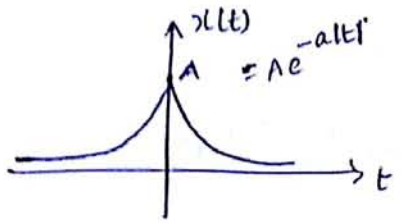
$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt = \int_{-\infty}^{0} A e^{at} \cdot e^{-j\Omega t} dt + \int_{0}^{\infty} A e^{-at} e^{-j\Omega t} dt$$

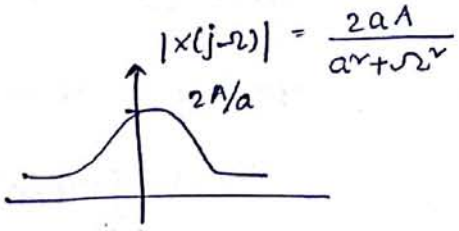
$$= \int_{-\infty}^{0} A \cdot e^{(a-j\Omega)t} dt + \int_{0}^{\infty} A e^{-(a+j\Omega)t} dt$$

$$= \left. \frac{A e^{(a-j\Omega)t}}{a-j\Omega} \right|_{-\infty}^{0} + \left. \frac{A e^{-(a+j\Omega)t}}{-(a+j\Omega)} \right|_{0}^{\infty}$$

$$= \frac{Ae^{0}}{a-j\Omega} - \frac{Ae^{-\infty}}{a-j\Omega} + \frac{Ae^{-\infty}}{-(a+j\Omega)} - \frac{Ae^{0}}{-(a+j\Omega)}$$

$$= \frac{A}{a-j\Omega} + \frac{A}{a+j\Omega} = \frac{2aA}{a^2+\Omega^2}$$





(c) Fourier Transform of a Constant :

Let $x(t) = A$ (constant)

If definition of F.T is directly applied, the constant will not satisfy the condition

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty.$$

Let $x_1(t)$ = double sided exponential sgl

$$x_1(t) = Ae^{-a|t|}$$

$$\therefore x(t) = A \lim_{a\to 0} x_1(t).$$

On taking F.T

$$F\{x(t)\} = F\{\lim_{a\to 0} x_1(t)\}$$

$$F\{x(t)\} = \lim_{a\to 0} F\{x_1(t)\}$$

$$X(j\Omega) = \lim_{a\to 0} [X_1(j\Omega)]$$

$$= \lim_{a\to 0} \frac{2aA}{\Omega^2+a^2} = 0 \text{ for all values of } \Omega \text{ except at } \Omega=0.$$

At $\Omega=0$, the above eqn represents an impulse of magnitude 'k'

$$\therefore X(j\Omega) = k\delta(\Omega) \quad ; \Omega=0$$
$$= 0 \quad ; \Omega\neq 0.$$

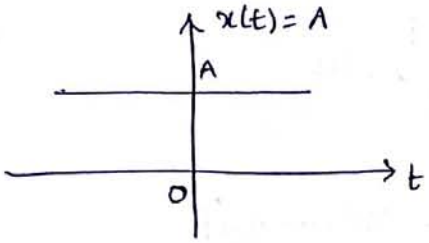
The magnitude 'k' can be evaluated as

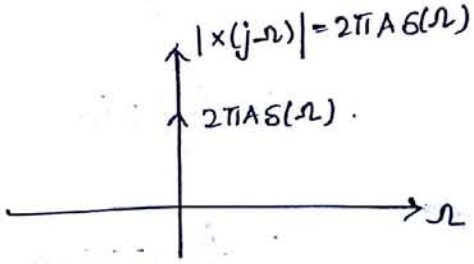
$$k = \int_{-\infty}^{\infty} \frac{2aA}{\Omega^2+a^2} d\Omega = 2aA\int_{-\infty}^{\infty} \frac{1}{\Omega^2+a^2} d\Omega \quad \left(\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} \right)$$

$$= 2aA\left[\frac{1}{a}\tan^{-1}\left(\frac{\Omega}{a}\right)\right]_{-\infty}^{\infty} = 2aA\left[\frac{1}{a}\tan^{-1}(+\infty) - \frac{1}{a}\tan^{-1}(-\infty)\right]$$

$$= 2aA\left[\frac{1}{a}\cdot\frac{\pi}{2} - \frac{1}{a}\left(-\frac{\pi}{2}\right)\right] = 2aA\left(\frac{\pi}{a}\right) = 2\pi A$$

$$\therefore F\{A\} = 2\pi A\,\delta(\Omega)$$





(d) Fourier transform of Unit step function:

$$u(t) = 1 \quad ; t \geq 0$$
$$= 0 \quad ; t < 0$$

$$\text{sgn}(t) = 2u(t) - 1 \Rightarrow u(t) = \frac{1}{2}\left[1 + \text{sgn}(t)\right]$$

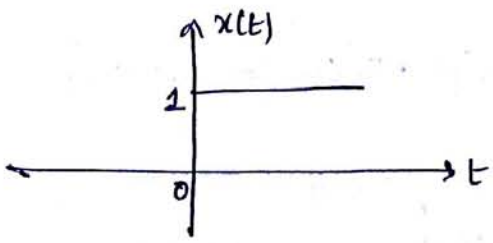
$$\therefore x(t) = u(t) = \frac{1}{2}\left[1 + \text{sgn}(t)\right]$$

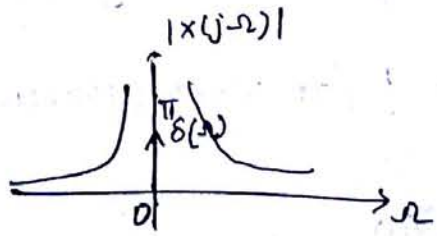
$$X(j\Omega) = F\left\{\frac{1}{2}\right\} + F\left\{\frac{1}{2}\text{sgn}(t)\right\}$$

$$= \frac{1}{2}F\{1\} + \frac{1}{2}F\{\text{sgn}(t)\}$$

$$= \frac{1}{2}\left[2\pi\delta(\Omega)\right] + \frac{1}{2}\left[\frac{2}{j\Omega}\right] \qquad \left(\because F\{\text{sgn}(t)\} = \frac{2}{j\Omega}\right)$$

$$F\{u(t)\} = \pi\delta(\Omega) + \frac{1}{j\Omega}$$





(e) Fourier transform of Sinusoidal signals:

$$x(t) = A\sin\Omega_0 t = \frac{A}{2j}\left[e^{j\Omega_0 t} - e^{-j\Omega_0 t}\right] \quad \left(\because \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}\right)$$

On taking F.T we get

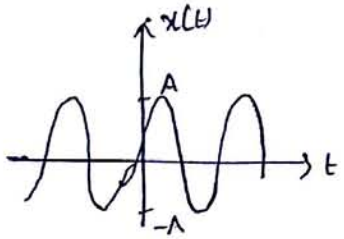
$$F\{x(t)\} = F\left\{\frac{A}{2j}\left(e^{j\Omega_0 t} - e^{-j\Omega_0 t}\right)\right\}$$

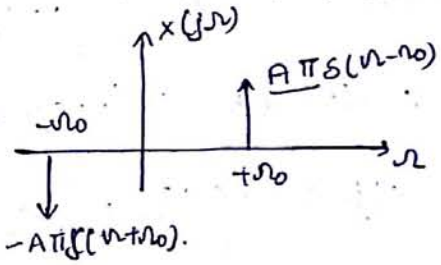
$$= \frac{A}{2j}\left[F\{e^{j\Omega_0 t}\} - F\{e^{-j\Omega_0 t}\}\right] \quad \left\{ \frac{1}{2}e^{j\omega_0 t} \xleftrightarrow{FT} \pi\delta(\omega-\omega_0)\right.$$

$$= \frac{A}{2j}\left[2\pi\delta(\Omega-\Omega_0) - 2\pi\delta(\Omega+\Omega_0)\right]$$

$$= \frac{A\pi}{j}\left[\delta(\Omega-\Omega_0) - \delta(\Omega+\Omega_0)\right]$$

$$\therefore F\{A\sin\Omega_0 t\} = \frac{A\pi}{j}\left[\delta(\Omega-\Omega_0) - \delta(\Omega+\Omega_0)\right]$$





(f) Fourier transform of cosinusoidal sgl:

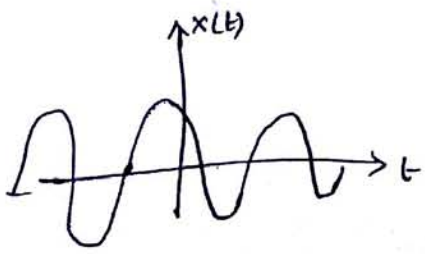
$$x(t) = A\cos\Omega_0 t = \frac{A}{2}\left[e^{j\Omega_0 t} + e^{-j\Omega_0 t}\right]$$

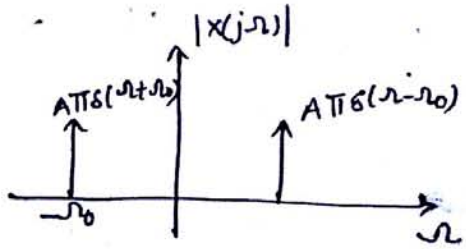
$\left\{ \because \text{we know that } 1 \xleftrightarrow{FT} 2\pi\delta(\omega),\ \frac{1}{2} \xleftrightarrow{FT} \pi\delta(\omega)\right.$

On taking F.T

$$F\{x(t)\} = F\left\{\frac{A}{2}\left(e^{j\Omega_0 t} + e^{-j\Omega_0 t}\right)\right\} = \frac{A}{2}\left\{F(e^{j\Omega_0 t}) + F(e^{-j\Omega_0 t})\right]$$

$$= \frac{A}{2}\left[2\pi\delta(\Omega-\Omega_0) + 2\pi\delta(\Omega+\Omega_0)\right] = A\pi\left[\delta(\Omega-\Omega_0) + \delta(\Omega+\Omega_0)\right]$$





Properties of F.T :

(1) Linearity : If $x(t) \xleftrightarrow{FT} X(\omega)$ & $y(t) \xleftrightarrow{FT} Y(\omega)$ then $z(t) = a\,x(t) + b\,y(t)$ $\xleftrightarrow{FT} Z(\omega) = a\,X(\omega) + b\,Y(\omega)$

proof

$$Z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} [a\,x(t) + b\,y(t)]\, e^{-j\omega t}\, dt$$

$$= a \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt + b \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= a\,X(\omega) + b\,Y(\omega).$$

(2) Time shift : If $x(t) \xleftrightarrow{FT} X(\omega)$ then $y(t) = x(t - t_0) \xleftrightarrow{FT} Y(\omega) = e^{-j\omega t_0} \cdot X(\omega)$

proof

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t - t_0)\, e^{-j\omega t}\, dt$$

put $t - t_0 = \tau \Rightarrow t = \tau + t_0$

$\therefore dt = d\tau$ and integration limits remains the same.

$$Y(\omega) = \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega(\tau + t_0)}\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau \cdot e^{-j\omega t_0}$$

$$Y(\omega) = e^{-j\omega t_0} \cdot X(\omega).$$

(3) Frequency shift : If $x(t) \xleftrightarrow{FT} X(\omega)$ then $y(t) = e^{j\omega_0 t} x(t) \xleftrightarrow{FT} Y(\omega) = X(\omega - \omega_0)$

proof:

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{j\omega_0 t} x(t) \cdot e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} x(t) \cdot e^{-jt(\omega - \omega_0)}\, dt$$

$$= X(\omega - \omega_0)$$

(4) Time Scaling :

If $x(t) \xleftrightarrow{FT} X(\omega)$ then $y(t) = x(at) \xleftrightarrow{FT} Y(\omega) = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$.

Meaning: Compression of a signal in time domain is equivalent to expansion in frequency domain & vice versa.

proof

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) \cdot e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$$

put $at = \tau \Rightarrow t = \tau/a$

$dt = \frac{d\tau}{a}$ and limits remain the same

$$\therefore \quad Y(\omega) = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega(\tau/a)} \cdot \frac{d\tau}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-j\left(\frac{\omega}{a}\right)\tau} d\tau$$

$$= \frac{1}{|a|} X\left(\frac{\omega}{a}\right).$$

(5) Frequency differentiation :

If $x(t) \xleftrightarrow{FT} X(\omega)$ then $-jt\, x(t) \xleftrightarrow{FT} \frac{d}{d\omega} X(\omega)$

Meaning: Differentiating the frequency spectrum is equivalent to multiplying the time domain signal by complex number $-jt$.

proof

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\frac{dX(\omega)}{d\omega} = \int_{-\infty}^{\infty} x(t) \cdot \frac{d}{d\omega}\left[e^{-j\omega t}\right] dt = \int_{-\infty}^{\infty} x(t) \cdot (-jt) e^{-j\omega t} dt$$

$$= -jt \cdot j \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt =$$

$$= \frac{1}{j} \int_{-\infty}^{\infty} (t x(t)) e^{-j\omega t} dt$$

$$= -j(t x(t))]$$

⑥ Time Differentiation:

If $x(t) \xleftrightarrow{FT} X(\omega)$ then $\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(\omega)$.

Meaning: Differentiation in time domain corresponds to multiplying by $j\omega$ in frequency domain.

Proof:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\cdot\frac{d}{dt}\left[e^{j\omega t}\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\cdot e^{j\omega t}\, d\omega\,(j\omega)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[j\omega X(\omega)\right] e^{j\omega t}\, d\omega$$

Thus fourier transform is multiplied by $j\omega$

(or)

$$F\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$F\left\{\frac{dx(t)}{dt}\right\} = \int_{-\infty}^{\infty}\left(\frac{dx(t)}{dt}\right) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t}\left(\frac{dx(t)}{dt}\right) dt$$

$$= \left[e^{-j\omega t}\cdot x(t)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(-j\omega)\, e^{-j\omega t}\cdot x(t)\, dt$$

$$= e^{-\infty}x(\infty) - e^{+\infty}x(-\infty) + j\omega\int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\, dt$$

$$\frac{dx(t)}{dt} = j\omega\int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\, dt$$

$$\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(\omega).$$

⑦ Time Integration:

If $F\{x(t)\} = X(j\omega)$ then $\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{FT} \frac{1}{j\omega} X(\omega)$

Proof

Let

$x(t)$ is expressed as

$$x(t) = \frac{d}{dt}\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right]$$

$$\therefore \quad F[x(t)] = F\left[\frac{d}{dt}\int_{-\infty}^{t} x(\tau)\,d\tau\right]$$

By differentiation property for right hand side

$$F[x(t)] = j\omega\, F\left\{\int_{-\infty}^{t} x(\tau)\,d\tau\right]$$

$$X(\omega) = j\omega\, F\left\{\int_{-\infty}^{t} x(\tau)\,d\tau\right]$$

$$\frac{1}{j\omega} X(\omega) = F\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right]$$

(or) $$F\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \frac{1}{j\omega} X(\omega).$$

⑧ Modulation or Multiplication:

If $x(t) \xleftrightarrow{FT} X(\omega)$ and $y(t) \xleftrightarrow{FT} Y(\omega)$ then $z(t) = x(t)\,y(t) \xleftrightarrow{FT}$

$$Z(\omega) = \frac{1}{2\pi}\left[X(\omega) * Y(\omega)\right]$$

Proof

$$Z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} x(t)\,y(t)\, e^{-j\omega t}\,dt \quad \longrightarrow ①$$

Inverse F.T states that

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\, e^{j\lambda t}\,d\lambda$$

putting for $x(t)$ in eq ①

$$Z(\omega) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) e^{j\lambda t} d\lambda \right] y(t) e^{-j\omega t} dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) \int_{-\infty}^{\infty} y(t) e^{-j(\omega-\lambda)t} dt \, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) Y(\omega-\lambda) \, d\lambda$$

$$= \frac{1}{2\pi} \left[X(\omega) * Y(\omega) \right]$$

Parseval's Theorem:

If $x(t) \xleftrightarrow{FT} X(\omega)$ then $E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$

$$= \int_{-\infty}^{\infty} |X(f)|^2 df.$$

proof

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot x^*(t) dt \longrightarrow ①$$

Inverse F.T states that

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Taking conjugate of both the sides

$$x^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} d\omega$$

putting above eqn $x^*(t)$ in eq ①

$$E = \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} \right] d\omega \cdot dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \cdot d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \cdot X(\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega.$$

Since $\omega = 2\pi f$, $d\omega = 2\pi\, df$

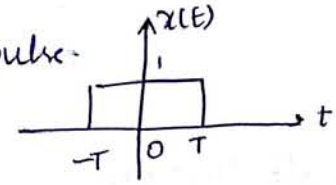
Equation becomes

$$E = \int_{-\infty}^{\infty} |X(f)|^2 df \cdot \frac{1}{2\pi} \cdot 2\pi$$

$$\boxed{E = \int_{-\infty}^{\infty} |X(f)|^2 df}$$

---

2) Obtain the fourier transform of rectangular pulse.



Sol

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-T}^{T} 1 \cdot e^{-j\omega t} dt = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T}^{T} = \left[\frac{e^{-j\omega T}}{-j\omega} - \frac{e^{+j\omega T}}{-j\omega}\right]$$

$$= -\frac{1}{j\omega}\left[e^{-j\omega T} - e^{j\omega T}\right]$$

$$= \frac{1}{\omega} \cdot \frac{2}{2j}\left[e^{j\omega T} - e^{-j\omega T}\right]$$

$$= \frac{2}{\omega}\left[\frac{e^{j\omega T} - e^{-j\omega T}}{2j}\right]$$

$$= \frac{2}{\omega} \sin(\omega T)$$

we know $\operatorname{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}$.

$\left\{ \because \frac{\sin\theta}{\theta} = \operatorname{sinc}\theta \right.$

$$X(\omega) = \frac{2}{\omega} \sin \omega T \cdot \frac{T}{T}$$

$$= 2T \cdot \frac{\sin \omega T}{\omega T} = 2T \operatorname{sinc}(\omega T)$$

→ Obtain the fourier transform of $x(t) = t e^{-at} u(t)$

Sol

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} t \cdot e^{-at} u(t) e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} t \cdot e^{-at} e^{-j\omega t} dt \quad \{ \because u(t) = 1 \text{ for } t \geq 0$$

$$X(\omega) = \int_{0}^{\infty} t \cdot e^{-(a+j\omega)t} dt$$

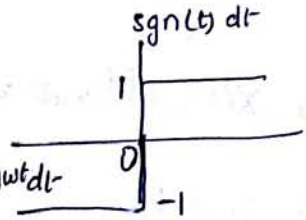
$$X(\omega) = \left[t \int e^{-(a+j\omega)t} dt - \int 1 \cdot \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} dt \right]_0^{\infty}$$

$$= \left[\frac{t \cdot e^{-(a+j\omega)t}}{-(a+j\omega)} + \frac{1}{a+j\omega} \cdot \frac{e^{-(a+j\omega)t}}{[-(a+j\omega)]} \right]_0^{\infty}$$

$$= 1/(a+j\omega)^2$$

$$\therefore t e^{-at} u(t) \xleftrightarrow{FT} \frac{1}{(a+j\omega)^2}$$

Signum function:



$\Rightarrow u(t) - u(-t)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt = \int_{-\infty}^{\infty} sgn(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} -1 \cdot e^{-j\omega t} dt + \int_{0}^{\infty} 1 \cdot e^{-j\omega t} dt$$

$$= \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-\infty}^{0} + \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{0}^{\infty} = \frac{1}{j\omega} \left[e^{-0} - e^{-\infty} \right] + \frac{1}{j\omega} [0 - 1]$$

$$= \frac{1}{j\omega} [1 - 0] - \frac{1}{j\omega} [0 - 1] = \frac{2}{j\omega}$$

(B) $x(t) = 1$

$$\int_{-\infty}^{\infty} |x(t)|\, dt = \int_{-\infty}^{\infty} dt \rightarrow \infty$$ i.e Dirichlet condition is not satisfied.

But its fourier transform can be calculated with the help of duality property.

$$\delta(t) \xleftrightarrow{FT} 1$$

Here $X(\omega) = 1$

$$\therefore \quad \delta(t) \xleftrightarrow{FT} X(\omega)$$

The duality property states that

$$x(t) \xleftrightarrow{FT} 2\pi\, x(-\omega)$$

Here $x(t) = 1$ then $x(-\omega)$ will be $\delta(-\omega)$.

$$1 \longleftrightarrow 2\pi\, \delta(-\omega)$$

Since it is impulse function $\delta(\omega)$ will be even functions of ω.

$$\delta(-\omega) = \delta(\omega)$$

$$1 \xleftrightarrow{FT} 2\pi\delta(\omega)$$

$\therefore$ if $x(t) = 1$, $X(\omega) = 2\pi\delta(\omega)$.

Duality property:

If $x(t) \xleftrightarrow{FT} X(\omega)$ then $X(t) \xleftrightarrow{FT} 2\pi\, x(-\omega)$

proof

Inverse F.T is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Interchanging 't' by 'ω'

$$x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t)\, e^{j\omega t}\, dt$$

Interchanging ω by $-\omega$

$$x(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \Rightarrow \boxed{X(t) = 2\pi\, x(-\omega)}$$

FOURIER TRANSFORM OF UNIT IMPULSE SIGNAL AND SIGNUM FUNCTION:

→ The impulse signal is defined as

$$x(t) = \delta(t) = \infty \; ; \; t = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

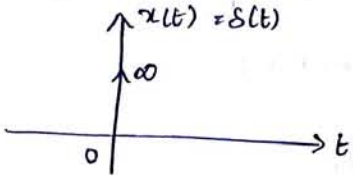
$$= 0 \; ; \; t \neq 0$$

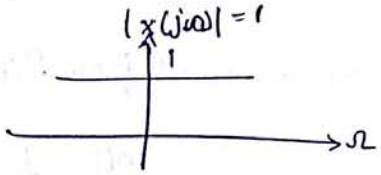
By definition of F.T

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt = \int_{-\infty}^{\infty} \delta(t) e^{-j\Omega t} dt$$

$$= 1 \times e^{-j\Omega t}\Big|_{t=0} = 1 \times e^{0} = 1$$

$$\therefore F\{x(t)\} = 1.$$





Signum function:

→ $x(t) = sgn(t) = 1 \quad ; \; t > 0$

$\qquad\qquad = -1 \quad ; \; t < 0$

$$\therefore \; sgn(t) = \underset{a \to 0}{Lt} \left[e^{-at} u(t) - e^{at} u(-t) \right]$$

$$x(t) = sgn(t) = \underset{a \to 0}{Lt} \left[e^{-at} u(t) - e^{at} u(-t) \right]$$

By definition of F.T

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt = \int_{-\infty}^{\infty} \underset{a \to 0}{Lt} \left[e^{-at} u(t) - e^{at} u(-t) \right] e^{-j\Omega t} dt$$

$$= \underset{a \to 0}{Lt} \left[\int_{0}^{\infty} e^{-at} \cdot e^{-j\Omega t} dt - \int_{-\infty}^{0} e^{at} e^{-j\Omega t} dt \right]$$

$$= \underset{a \to 0}{Lt} \left[\int_{0}^{\infty} e^{-(a+j\Omega)t} dt - \int_{-\infty}^{0} e^{(a-j\Omega)t} dt \right]$$

$$= \underset{a \to 0}{Lt} \left[\left[\frac{e^{-(a+j\Omega)t}}{-(a+j\Omega)} \right]_{0}^{\infty} - \left[\frac{e^{(a-j\Omega)t}}{a-j\Omega} \right]_{-\infty}^{0} \right]$$

$$= \underset{a\to 0}{Lt}\left[\frac{e^{-\infty}}{-(a+j\Omega)} - \frac{e^{0}}{-(a+j\Omega)} - \frac{e^{0}}{a-j\Omega} + \frac{e^{-\infty}}{a-j\Omega}\right]$$

$$= \underset{a\to 0}{Lt}\left[\frac{1}{a+j\Omega} - \frac{1}{a-j\Omega}\right] = \frac{1}{j\Omega} + \frac{1}{j\Omega} = \frac{2}{j\Omega}$$

INTRODUCTION TO HILBERT TRANSFORM:

The hilbert transform of $h(t)$ is defined by

$$\hat{h}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{h(\tau)}{t-\tau}\,d\tau$$

The inverse hilbert transform by means of which the original signal $h(t)$ is recovered from $\hat{h}(t)$ is defined by

$$h(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\hat{h}(\tau)}{t-\tau}\,d\tau$$

→ The functions $h(t)$ and $\hat{h}(t)$ are said to constitute a hilbert transform pair.

APPLICATIONS:

→ Areas of signal processing, analysis and synthesis of signals, design of filters etc.

→ To represent band pass signals

→ To realize phase selectivity in the generation of SSB systems.

→ To relate the gain & phase characteristics of linear communication channels & filters of minimum phase type.

SAMPLING:

→ It is a process of converting a continuous time signal into discrete-time signal.

→ The time interval between two subsequent sampling instants is called sampling interval.

→ Only we need to consider that the signal sampling rate must be kept high in order to reconstruct the original signal from its samples.

SAMPLING THEOREM - GRAPHICAL AND ANALYTICAL PROOF FOR BAND LIMITED SIGNALS:

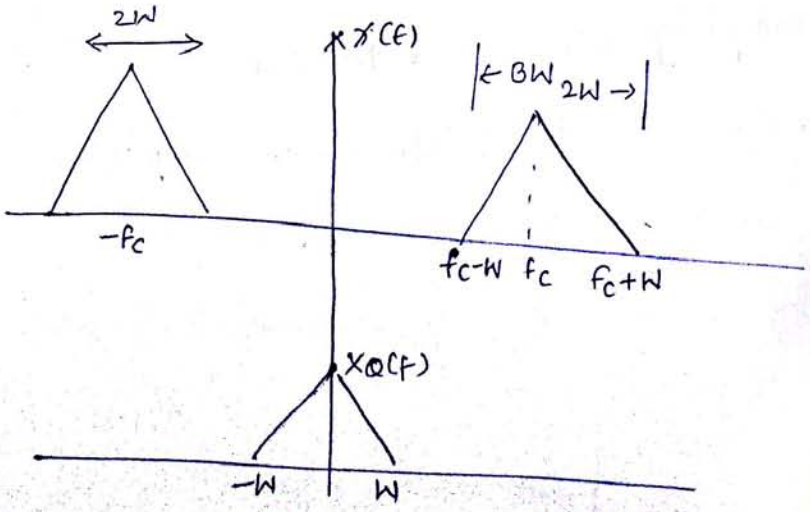
→ A band limited signal $x(t)$ with $X(j\Omega) = 0$ for $|\Omega| > \Omega_m$ is uniquely determined from its samples $x(nT)$, if the sampling frequency $f_s \geq 2f_m$ i.e sampling frequency must be at least twice the highest frequency present in the signal.

→ The time interval between successive samples is called sampling period and its reciprocal $1/T = f_s$ is called sampling rate.

proof:

Let the signal $x(t)$ be a bandpass nature. The bandpass has the bandwidth of '2W' centered around f_c. The bandpass signal is represented interms of its inphase and quadrature components as

$$x(t) = \underbrace{x_I(t)}_{\text{phase}} \cos(2\pi f_c t) - \underbrace{x_Q(t)}_{\text{quadrature}} \sin(2\pi f_c t)$$



→ The inphase and quadrature components are sampled at $f_s = 4W$ and obtained by multiplying $x(t)$ by $\cos\omega_c t$ and $\sin\omega_c t$.

→ Suppressing the sum frequencies using LPF.

→ Thus $x_I(t)$ and $x_Q(t)$ contain only low frequency components.

→ The spectrum of these components is limited b/n $-f_m$ to f_m as shown in the figure above.

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{4f_m}\right) \operatorname{sinc}\left[2f_m t - \frac{n}{2}\right] \cos\left(\omega_c\left(t - \frac{n}{4f_m}\right)\right)$$

compare reconstruction formula with that of LP signals given in the interpolation formula, we observe $x(t)$ is replaced by $x\left(\frac{n}{4f_m}\right)$.

$$x\left(\frac{n}{4f_m}\right) = x(nT_s) \rightarrow \text{sampled version of BP signal}$$

$$T_s = 1/4f_m$$

$4f_m \rightarrow$ samples/sec are taken.

→ Band pass signal $BW = 2f_m$ can be completely recovered by its samples.

→ Minimum sampling rate = twice the BW

$$f_s = 2 \times 2f_m = 4f_m \text{ samples/sec.}$$

SAMPLING THEOREM FOR LOW PASS SIGNAL:

→ A LP signal contains frequencies from 1Hz to some higher value.

statements:

1. A band limited signal of finite energy, which has no frequency components higher than W Hertz, is completely described by specifying the values of the signals at instants of time separated by $\frac{1}{2W}$ seconds.

2. A band limited signal of finite energy, which has no freq components higher than W hertz may be completely recovered from the knowledge of its samples taken at the rate 2W samples per second → reconstruction of sgl.

3. A CT signal can be completely represented in its samples and recovered back iff the sampling frequency is twice of the highest frequency content of the signal i.e $f_s > 2W$ → higher frequency.
   ↳ sampling frequency

Proof: There are 2 parts
1. Representation of $x(t)$ in terms of samples
2. Reconstruction of $x(t)$ from its samples.

1. Representation of $x(t)$ in its samples $x(nT_s)$:

steps:
1. Define $x_\delta(t)$
2. F.T of $x_\delta(t)$ i.e $\delta(f)$
3. Relation b/n $X(f)$ and $X_\delta(f)$
4. Relation b/n $x(t)$ & $x(nT_s)$

1. Define $x_\delta(t)$: $x_\delta(t) = \sum_{n=-\infty}^{\infty} x(t) \cdot \delta(t - nT_s) \rightarrow (1)$ it is product of x_δ and impulse train $\delta(t)$ as shown in fig (1)

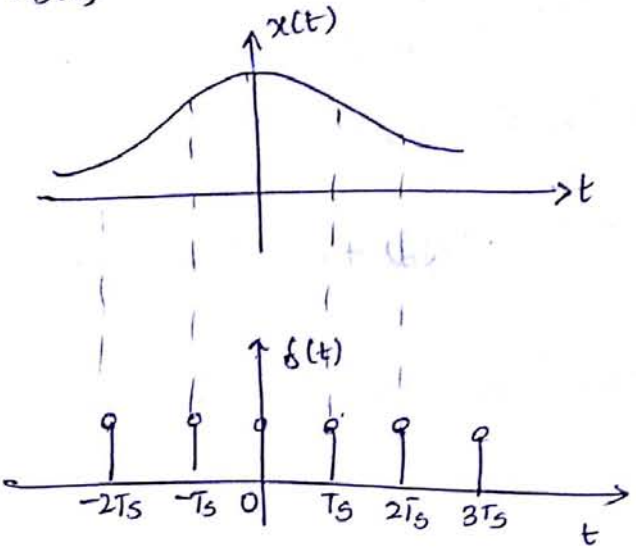
in (1) ⇒ $\delta(t - nT_s)$ indicates the samples placed at $\pm T_s, \pm 2T_s$.

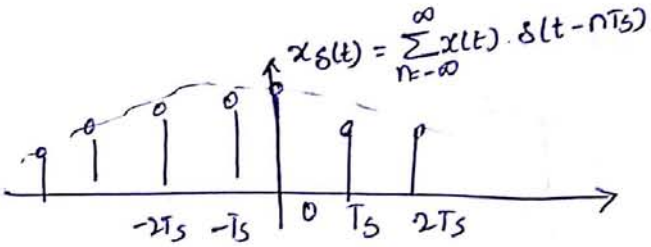
2. F.T of $x_\delta(t)$:

$$X_\delta(t) = FT\left[\sum_{n=-\infty}^{\infty} x(t)\,\delta(t - nT_s)\right]$$

Proof:

step 1: Define $x_\delta(t)$





$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(t).\,\delta(t-nT_s)$$

↓

product of $x(t)$ and impulse train. $\delta(t-nT_s)$ indicates samples placed at $\pm T_s, \pm 2T_s + \cdots$

step 2: Fourier transform of $x_\delta(t)$ i.e $X_\delta(f)$.

$$X_\delta(f) = FT\left\{\sum_{n=-\infty}^{\infty} x(t)\,\delta(t-nT_s)\right] \longrightarrow (2)$$

We know that fourier transform of product or multiplication in time domain is the convolution in frequency domain.

$$X_\delta(f) = FT\{x(t)\} * FT\{\delta(t-nT_s)\} \longrightarrow (1)$$

By definition $x(t) \longleftrightarrow X(f)$ and $\delta(t-nT_s) \xleftrightarrow{FT} f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s)$

∴ Eq (1) becomes

$$X_\delta(f) = X(f) * f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s)$$

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f) * \delta(f - nf_s)$$

By shifting property of impulse function

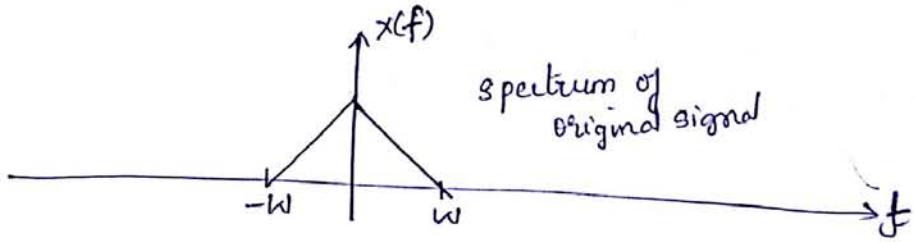
$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

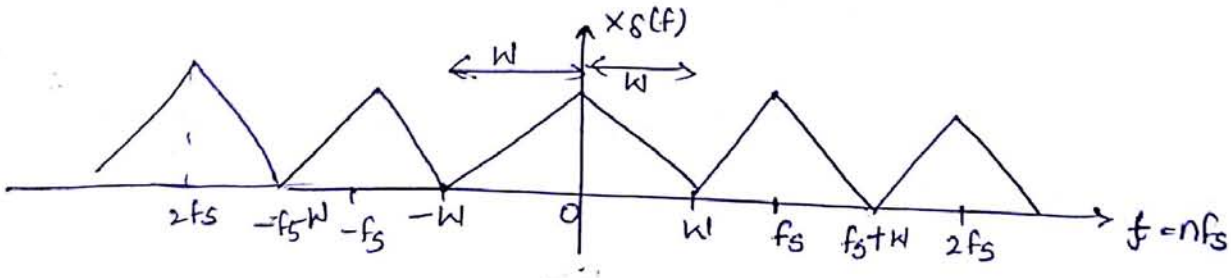
$$= \cdots f_s X(f+2f_s) + f_s X(f+f_s) + f_s X(f) + f_s X(f-f_s) + \cdots$$

$\therefore$ (i) The RHS of the above eqn is placed at i.e $X(f)$ is placed at $\pm f_s, \pm 2f_s, \pm 3f_s \cdots$

(ii) $X(f)$ is periodic in f_s.

(iii) If $f_s = 2W$, then spectrums $X(f)$ just touch each other





Step 3: Relation between $X(f)$ and $X_\delta(f)$.

Let us assume that $f_s = 2W$ as per above diagram.

$$X_\delta(f) = f_s \cdot X(f) \qquad -W \le f \le W \;\&\; f_s = 2W$$

$$X(f) = \frac{1}{f_s} \cdot X_\delta(f) \longrightarrow (1)$$

Step 4: Relation between $x(t)$ and $x(nT_s)$

F.T $\quad X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

$$X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n}$$ → frequency of DT.

If we replace $X(f)$ by $X_\delta(f)$ then 'f' becomes frequency of CT signal

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-2\pi f/f_s \cdot n}$$

→ 'f' is frequency of CT signal & f/f_s = frequency of DT signal

Since $x(n) = x(nT_s)$

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s} \longrightarrow (2)$$

putting above eq (2) in eq (1)

$$X(f) = \frac{1}{f_s} \cdot \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s}.$$

Inverse F.T of above equation gives $x(t)$

$$x(t) = IFT\left\{ \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s} \right\}$$

Note:

(1) $x(t)$ is represented completely in terms of $x(nT_s)$

(3) Above eqn holds for $f_s = 2W$ .ie samples are taken at the rate of 2W or higher, $x(t)$ is completely represented by samples.

Part II: Reconstruction of $x(t)$ from its samples.

step 1: The IFT of $x(t) = \int_{-\infty}^{\infty} \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s} \Big\} e^{j2\pi f t}\, df$

here integration can be taken from $-W \le f \le W$. Since $X(f) = \frac{1}{f_s} X_\delta(f)$ for $-W \le f \le W$

$$x(t) = \int_{-W}^{W} \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s} \cdot e^{j2\pi f t}\, dt$$

Interchanging the order of summation & integration

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{1}{f_s} \int_{-W}^{W} e^{j2\pi f (t - nT_s)}\, df$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{1}{f_s} \left[\frac{e^{j2\pi f(t-nT_s)}}{j2\pi(t-nT_s)} \right]_{-W}^{W}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{1}{f_s} \left\{ \frac{e^{j2\pi W(t-nT_s)} - e^{-j2\pi W(t-nT_s)}}{j2\pi(t-nT_s)} \right\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{1}{f_s} \frac{\sin 2\pi W(t-nT_s)}{\pi(t-nT_s)}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{\sin \pi(2Wt - 2WnT_s)}{\pi(f_s t - nT_s \cdot f_s)}$$

Here $f_s = 2W$ hence $T_s = \frac{1}{f_s} = \frac{1}{2W}$

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \operatorname{sinc}(2Wt-n) \quad \left\{ \because \frac{\sin \pi\theta}{\pi\theta} = \operatorname{sinc} \theta \right\}$$

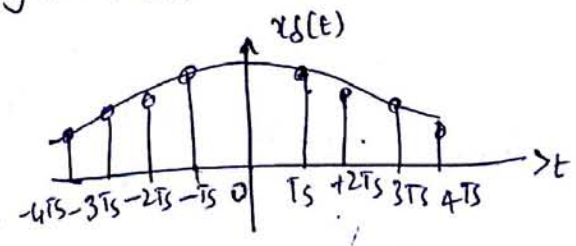
$$x(t) = \cdots + x(-2T_s)\operatorname{sinc}(2Wt+2) + x(-T_s)\sin(2WT+1)$$
$$+ x(0)\operatorname{sinc}(2Wt) + x(T_s)\sin(2Wt-1) \cdots$$

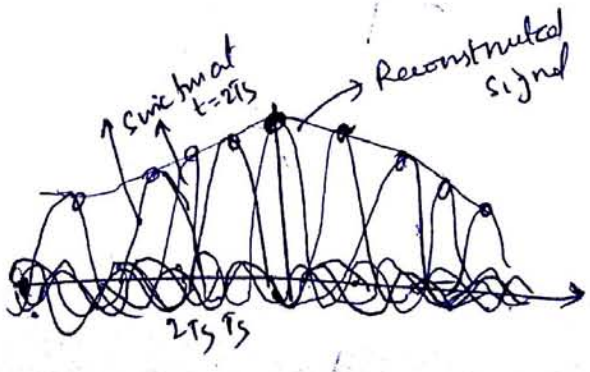
(i) The samples $x(nT_s)$ are weighted by sinc functions

(ii) The sinc function is the interpolating function.

Step 3: Reconstruction of $x(t)$ by LPF

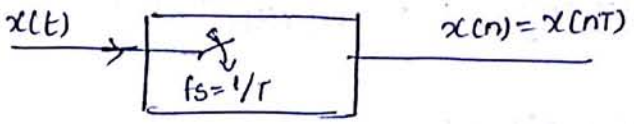
When the interpolated signal is passed through low pass filter of bandwidth $-W \leq f \leq W$ then the reconstructed waveform is obtained as shown in figure.



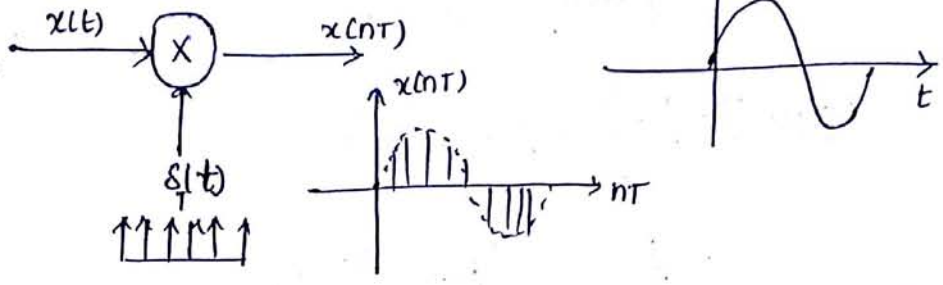


TYPES OF SAMPLING:

Impulse Sampling:



(a) ficitious switch.



→ The switch is closed for short interval of time τ, during which the signal is available at the output.

→ If input is $x(t)$ & output is $x(nT), n=0,\pm1,\pm2\ldots$ i.e $x(nT)$ is called sample sequence of $x(t)$ where T is time interval b/w successive samples and sampling frequency $f_s = 1/T$ HZ.

→ $$x_s(t) = x(t)\,\delta_T(t)$$

where $$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT).$$

$$x_{(s)}(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

Applying Fourier transform

$$F[x_s(t)] = F\left[\sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)\right]$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\,F[\delta(t-nT)]$$

we have $F[\delta(t-nT)] = \int_{-\infty}^{\infty} \delta(t-nT)\,e^{-j\omega t}\,dt = e^{-j\omega nT}$

but $\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) = \frac{1}{T}\sum_{m=-\infty}^{\infty} e^{jm\omega_0 t}$.

$$x_s(t) = x(t)\left[\frac{1}{T}\sum_{m=-\infty}^{\infty} e^{jm\Omega_0 t}\right] = \frac{1}{T}\sum_{m=-\infty}^{\infty} x(t)e^{jm\Omega_0 t}$$

$$F[x_s(t)] = \frac{1}{T}\sum_{m=-\infty}^{\infty} F\left[x(t)e^{jm\Omega_0 t}\right]$$

If $F[x(t)] = X(j\Omega)$ then

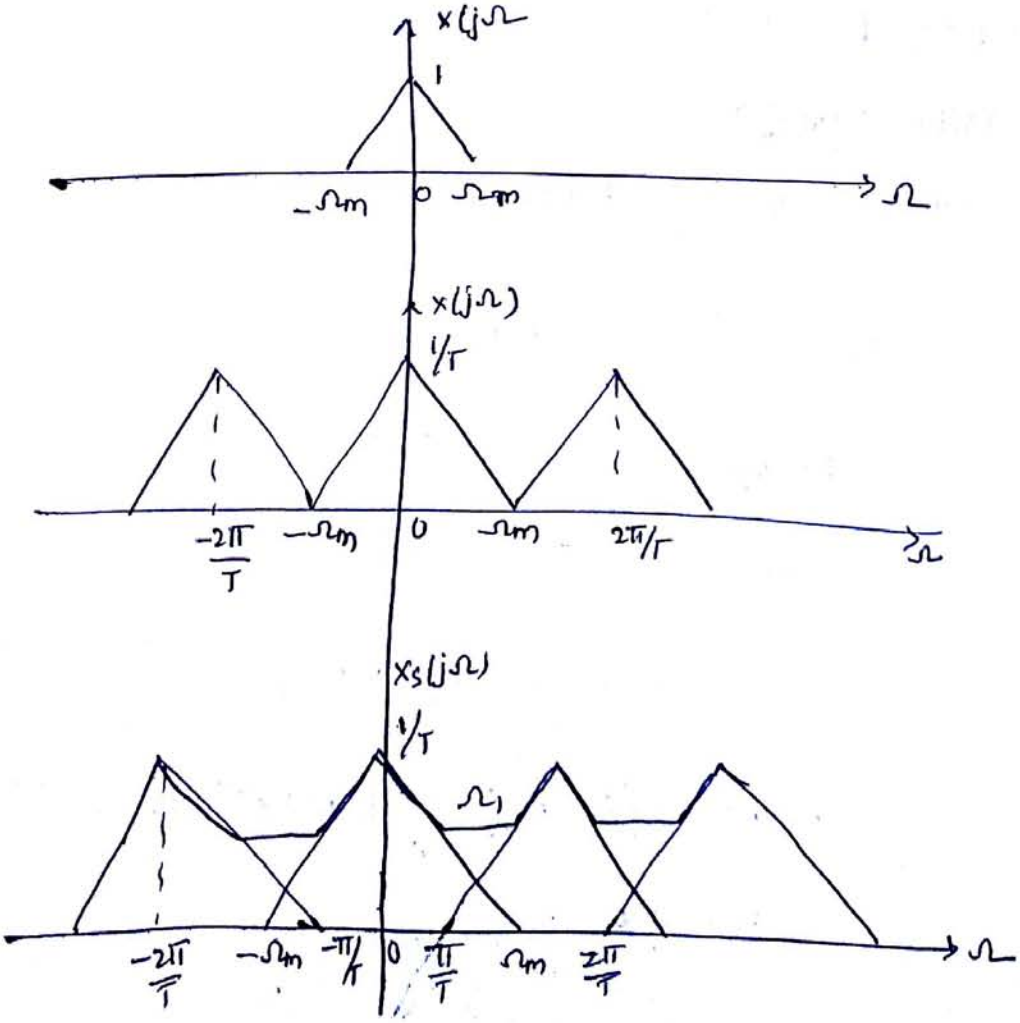
$$F\left[x(t)e^{jm\Omega_0 t}\right] = X(j(\Omega - m\Omega_0))$$

$$F[x_s(t)] = \frac{1}{T}\sum_{m=-\infty}^{\infty} X(j(\Omega - m\Omega_0))$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty} X\left(j\left(\Omega - \frac{2\pi m}{T}\right)\right).$$

→ Now consider a signal $x(t)$ band limited to f_m. That is the highest frequency component of $x(t)$ is f_m.

$$X(j\Omega) = 0 \text{ for } |\Omega| > \Omega_m.$$



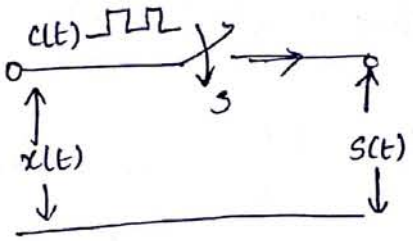
Natural Sampling : (Chopper sampling)

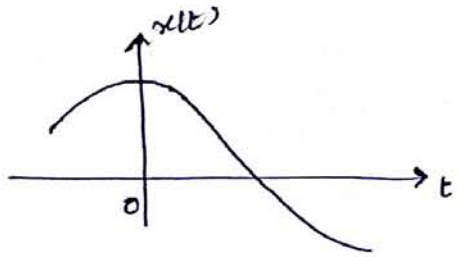
In natural sampling the pulse has a finite width, τ. It is also called as chopper sampling because the waveform of the sampled signal appears to be chopped off from the original signal waveform.

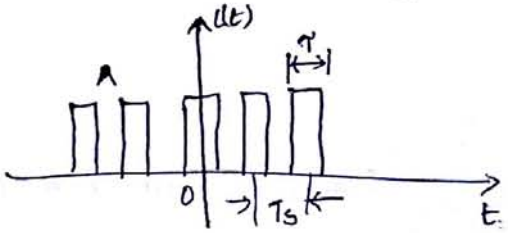
→ Let $x(t)$ be analog continuous time signal that has to be sampled at the rate of f_s Hz.

→ A sampled signal $s(t)$ is obtained by multiplication of a sampling function and signal $x(t)$.

→ Sampling function is denoted by $c(t)$ with pulses of width τ and frequency equal to f_s.





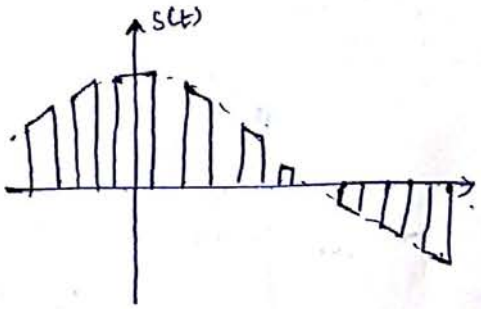


→ When $c(t)$ goes high, a switch 's' is closed.

$s(t) = x(t)$ when $c(t) = A$

$s(t) = 0$ when $c(t) = 0$

$$s(t) = c(t) \cdot x(t) \rightarrow (1)$$



Exponential Fourier series for periodic waveform is given as

$$x(t) = \sum_{n=-\infty}^{\infty} x(n) e^{jn\omega_0 t}$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(n) e^{jn2\pi t/T_0} \rightarrow (2)$$

For periodic pulse train of $c(t)$

$$T_0 = T_s = \frac{1}{f_s}$$

(or) $f_0 = f_s = \frac{1}{T_0} = \frac{1}{T_s}$ = frequency of $c(t)$.

Equation (2) will be [with $x(t) = c(t)$]

$$c(t) = \sum_{n=-\infty}^{\infty} c(n)\, e^{j2\pi n t \cdot f_s} \longrightarrow (3)$$

rectangular pulse train.

$$c_n = \frac{TA}{T_0} \operatorname{sinc}(f_n T) \quad \text{and } T = \text{pulse width} = \tau \longrightarrow (4)$$

$\hookrightarrow f_n = n f_s$

Harmonic frequency

putting eq (4) in (3)

$$c(t) = \sum_{n=-\infty}^{\infty} \frac{TA}{T_s} \operatorname{sinc}(f_n \tau)\, e^{j2\pi f_s n t} \longrightarrow (5)$$

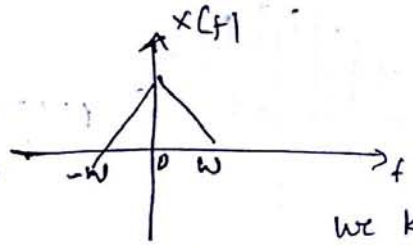
Substituting in eq (1), (5) we have

$$s(t) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} \operatorname{sinc}(f_n \tau)\, e^{j2\pi f_s n t} \cdot x(t) \longrightarrow (6)$$

Represents naturally sampled signals.

Fourier transform of $s(t)$ is obtained

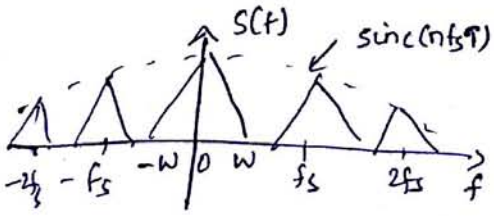
$$S(f) = FT\{s(t)\}$$



$$= \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} \operatorname{sinc}(f_n \tau)\, FT\left\{ e^{j2\pi f_s n t} \cdot x(t) \right\}$$

we know frequency shifting property

$$e^{j2\pi f_s n t} \cdot x(t) \longleftrightarrow X(f - f_s n)$$



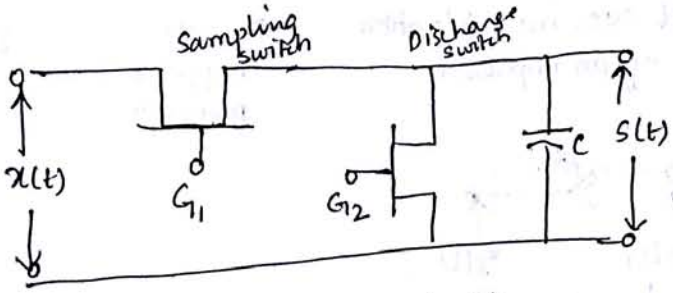
$$S(f) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} \operatorname{sinc}(f_n \tau)\, X(f - f_s n).$$

spectrum of Naturally sampled sgl.

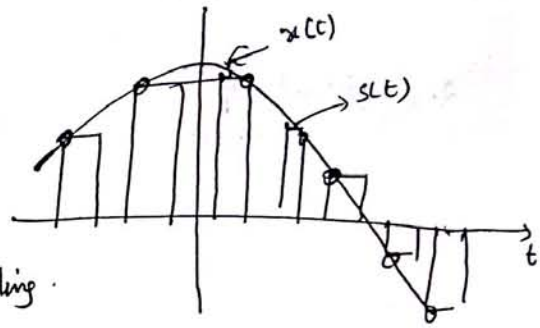
Flat top Sampling or Rectangular Pulse Sampling:

→ This is also practically possible method but easy method to get flat top samples.

→

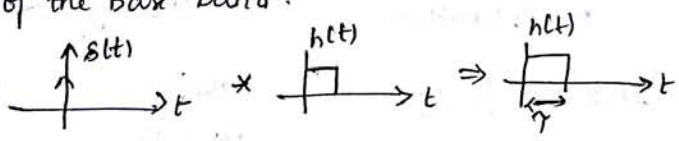


↓ sample & hold circuit generating flat top sampling.



→ The width of the pulse in flat top sampling and natural sampling is increased to reduce the transmission bandwidth.

→ Here the top of the samples remain const which is equal to instantaneous value of the base band.



$\therefore\ x(t) * \delta(t) = x(t)$

$h(t) * \delta(t) = h(t)$.

→ $$y(t) = s(t) * h(t) = h(t) * \left[x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right] \rightarrow (1)$$

Applying F.T

$$Y(f) = H(f) \left[X(f) \cdot \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \right] \rightarrow (2)$$

↓ sinc fn.

→ We can say that the primary effect of flat top sampling is an attenuation of high frequency components. This effect is called aperature effect.

→ This can be compensated by an equalizing filter with transfer fn

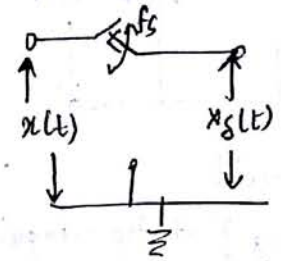
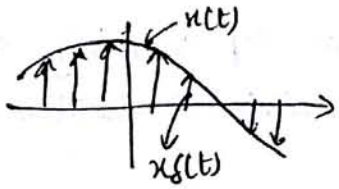
$$H_{eq}(f) = \frac{1}{|H(f)|} \rightarrow (3)$$

if $\tau \ll T_s$ then $H(f)$ will essentially be constant over the based band.

$$\therefore\ Y(f) = H(f)\, f_s \sum_{k=-\infty}^{\infty} X(f - kf_s)$$

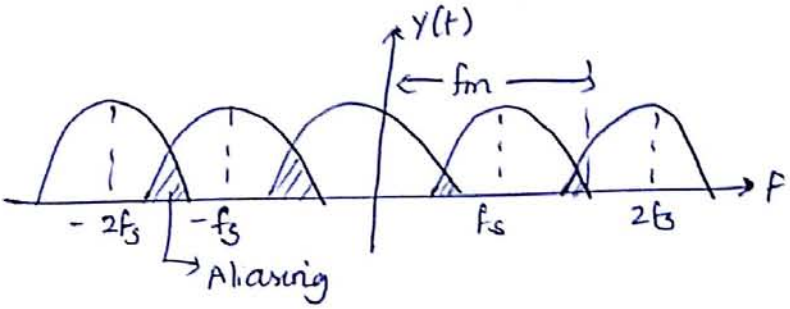
where $H(f) = \tau\, \mathrm{sinc}(f\tau)\, e^{-j2\pi f \tau} \rightarrow (5)$

Comparision of Various Sampling Techniques

S.No	Parameter of comparision	Ideal or instantaneous sampling	Natural sampling	Flat top sampling
1	Principle of sampling	It uses multiplication by an impulse fn	It uses chopping principle	It uses sample and hold circuit
2.	Circuit of sampler			
3.	Waveforms			
4.	Realizability	This is not pratically possible method	Used Practically	Used practically
5.	Sampling rate	tends to infinity	satisfies nyquist rate	satisfies nyquist criteria
6.	Noise interference	Noise interference is maximum	is minimum	is maximum
7.	Time domain representation	$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)$	$s(t) = \frac{\pi A}{T_s}\sum_{n=-\infty}^{\infty} x(t)\sin(n f_s T)$ $e^{j2\pi n f_s t}$	$s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)$ $\delta(t-nT_s)$
8.	Frequency domain representation	$X_\delta(t) = f_s\sum_{n=-\infty}^{\infty} X(f-nf_s)$	$s(f) = \frac{\pi A}{T_s}\sum_{n=-\infty}^{\infty} \mathrm{sinc}(n f_s \pi)$ $X(f-nf_s)$	$s(f) = f_s\sum_{n=-\infty}^{\infty}$ $X(f-nf_s)H(f)$

EFFECTS OF UNDERSAMPLING - ALIASING:

→ When a continuous time band-limited signal is sampled at a rate lower than nyquist rate, $f_s < 2f_m$ then successive cycles of the spectrum $Y(f)$ of the sampled signal $y(t)$ overlap with each other as shown in below



→ Hence the signal is undersampled when $f_s < 2f_m$ and some amount of aliasing is produced.

→ Aliasing is the phenomenon in which a high freq component in the frequency spectrum of signal takes identity of a LF component in the spectrum of sampled signal.

→ Due to aliasing, it is not possible to recover the original signal $x(t)$ from sampled signal $y(t)$ by LPF.

→ The signal is distorted and the information signal contains a large no. of frequencies dividing the sampling frequency is always a problem.

→ The signal is 1st passed through an LPF which blocks all the freqs above f_m Hz → band limiting of original signal.

→ This LPF is called pre-alias filter, it is used to prevent aliasing effect.

To avoid aliasing

(1) pre-alias filter must be used to limit the band of frequencies of the signal to f_m Hz

(2) Sampling frequency f_s must be selected such that $f_s \geq 2f_m$.

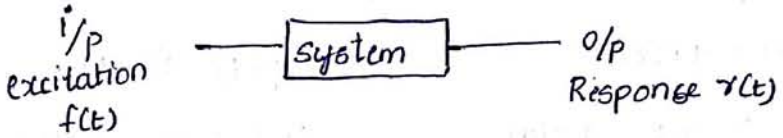
UNIT-3

SIGNAL TRANSMISSION THROUGH LINEAR SYSTEMS

System: A system is defined as set of rules that associates an o/p time function to every i/p time function.

(or)

A system is an interconnection of elements which produces expected o/p for available i/p.



$f(t) \longrightarrow r(t)$

→ System is an mathematical operator which maps i/p into o/p

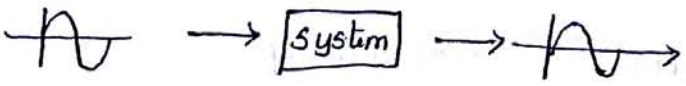
Classification of system

- continuous time system
- Discrete time system

1. Static & Dynamic systems
2. Linear & Non-Linear
3. Time invariant & Time variant
4. Linear TI & LTIV
5. Stable system
6. Casual & non-causal systems.

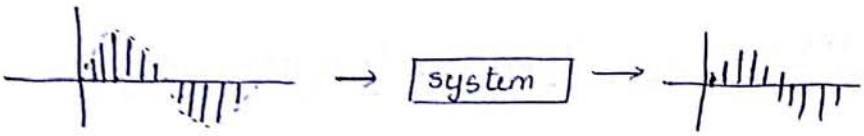
(i) continuous time systems

→ A continuous time system operates on a continuous time i/p signal to produce a continuous time o/p signal



(ii) Discrete time systems:

A discrete time system operates on a discrete time i/p signal to produce a discrete time o/p signal.



Classifications :

(i) Static and Dynamic Systems:

→ A static system or system is said to be static if its o/p at any instant depends only on present values of i/p.

Ex: $y(t) = ax(t)$

at $t=0$ $y(0) = ax(0)$

at $t=1$ $y(1) = ax(1)$

(ii) $y(t) = a^2 x(t)$

at $t=0$ $y(0) = a^2 x(0)$

at $t=1$ $y(1) = a^2 x(1)$

→ A system is said to be dynamic if its o/p depends on present & past values of i/p.

Ex: $y(t) = x(t-1) + x(t-2) + x(t)$

at $t=2$

$$y(2) = x(2-1) + x(2-2) + x(2) = \underbrace{x(1) + x(0)}_{\text{past}} + \underbrace{x(2)}_{\text{present}}$$

(ii) Linear and Non Linear Systems :

→ A system is said to be linear if its satisfies the superposition principle.

→ It states that the response of the system to a weighted sum of signals be equal to the corresponding weighted sum of o/p's of the system to each of the individual i/p signal.

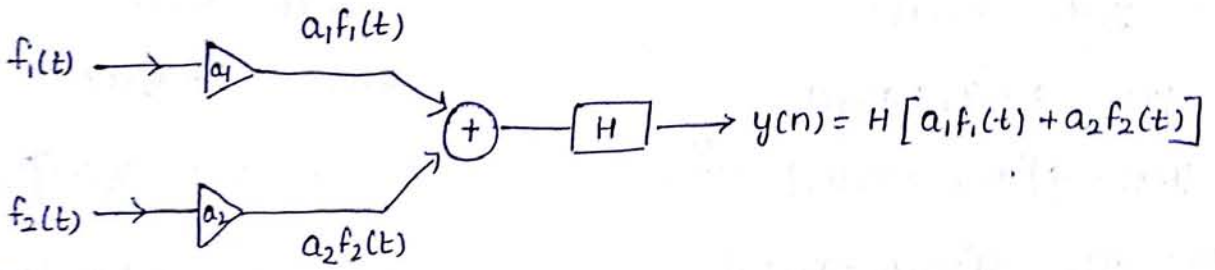
$$H[a_1 f_1(t) + a_2 f_2(t)] = a_1 H[f_1(t)] + a_2 H[f_2(t)]$$

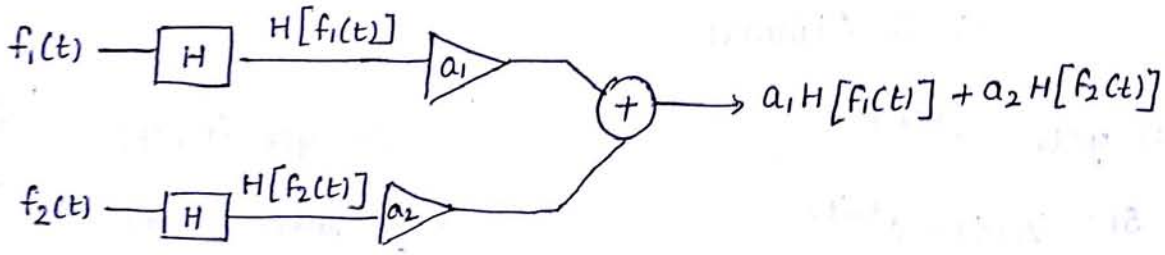
where a_1, a_2 are weighted constants.

$$a_1 f_1(t) \xrightarrow{\text{Response}} a_1 H[f_1(t)]$$

$$a_2 f_2(t) \xrightarrow{\text{Response}} a_2 H[f_2(t)]$$

$$H[a_1 f_1(t) + a_2 f_2(t)] \longrightarrow a_1 H[f_1(t)] + a_2 H[f_2(t)]$$





Block diagram.

→ Any system which does not obey the above principle is called as non-linear systems.

Check for Linearity:

Procedure:

1. Apply different i/p's separately and get the o/p.
2. Apply different i/p's simultaneously and get the output.
3. If both outputs are same it is linear otherwise non-linear.

Ex:

(i) $y(t) = 4 \sin t\, x(t)$

Step 1: $y_1(t) = 4 \sin t\; x_1(t)$

$y_2(t) = 4 \sin t\; x_2(t)$

$y_1(t) + y_2(t) = 4 \sin t\, [x_1(t) + x_2(t)]$

Step 2: $y(t) = 4 \sin t\, [x_1(t) + x_2(t)]$

$S_1 = S_2$

(2) $y(t) = a x(t)$

Sol S_1: $y_1(t) = a\, x_1(t)$

$y_2(t) = a\, x_2(t)$

$y(t) = a\, x_1(t) + a x_2(t)$

$y(t) = a[x_1(t) + x_2(t)]$

S_2: $y(t) = a[x_1(t) + x_2(t)]$

$S_1 = S_2$. (Linear)

(4) $y(t) = e^{x(t)}$

Sol S_1: $y_1(t) = e^{x_1(t)}$

$y_2(t) = e^{x_2(t)}$

$y(t) = e^{x_1(t)} + e^{x_2(t)}$

$y(t) = e^{[x_1(t) + x_2(t)]}$

S_2: $e^{x_1(t)} \cdot e^{x_2(t)}$

$S_1 \neq S_2$ (Non-Linear)

(6) $y(t) = x(t - t_0)$

Sol S_1: $y_1(t) = x_1(t - t_0)$

$y_2(t) = x_2(t - t_0)$

$y(t) = x_1(t - t_0) + x_2(t - t_0)$

S_2: $y(t) = x_1(t - t_0) + x_2(t - t_0)$

$S_1 = S_2$ (Linear

(8) $y(t) = x(t+1) e^{-t}$

Sol S_1: $y_1(t) = x_1(t+1) e^{-t}$; $y_2(t) = x_2(t+1) e^{-t}$

$y(t) = e^{-t}[x_1(t+1) + x_2(t+1)]$

S_2: $y(t) = e^{-t}[x_1(t+1) + x_2(t+1)]$

$S_1 = S_2$ (Linear)

(9) $y(t) = 4x(t) + 2\dfrac{dx(t)}{dt}$ → Linear.

(3) $y(t) = x^2(t)$

Sol S_1: $y_1(t) = x_1^2(t)$

$y_2(t) = x_2^2(t)$

$y(t) = x_1^2(t) + x_2^2(t)$

S_2: $y(t) = [x_1(t) + x_2(t)]^2$ →

$S_1 \neq S_2$. (Non-Linear)

(5) $y(t) = t\, x(t)$

Sol $y_1(t) = t\, x_1(t)$

S_1: $y_2(t) = t\, x_2(t)$

$y(t) = t[x_1(t) + x_2(t)]$

S_2: $y(t) = t[x_1(t) + x_2(t)]$

$S_1 = S_2$ (Linear)

(11) $y(t) = x(t^2)$

Sol S_1: $y_1(t) = x_1(t^2)$

$y_2(t) = x_2(t^2)$

$y(t) = x_1(t^2) + x_2(t^2)$

S_2: $y(t) = x_1(t^2) + x_2(t^2)$

$S_1 = S_2$

(Linear)

(7) $y(t) = 3x(t+3)$

Sol S_1: $y_1(t) = 3x_1(t+3)$

$y_2(t) = 3x_2(t+3)$

$y(t) = 3[x_1(t+3) + x_2(t+3)]$

S_2: $3[x_1(t+3) + x_2(t+3)]$

$S_1 = S_2$ (Linear)

(12) $y(t) = Ax(t) + B$

Sol $y_1(t) = Ax_1(t) + B$

$y_2(t) = Ax_2(t) + B$

$y(t) = A[x_1(t) + x_2(t)] + 2B$

$y(t) = A[x_1(t) + x_2(t)] + B$

$S_1 \neq S_2$ = Non-Linear

(9) $y(t) = \cos[x(t)]$

Sol S_1: $y_1(t) = \cos[x_1(t)]$; $y_2(t) = \cos[x_2(t)]$

$y(t) = \cos[x_1(t)] + \cos[x_2(t)]$

S_2: $\cos[x_1(t) + x_2(t)]$

$S_1 \neq S_2$ (Non-Linear)

(10) $y(t) = k\Delta x(t)$ where $\Delta x(t) = [x(t+1) - x(t)]$

Sol S_1: $y_1(t) = k[x_1(t+1) - x_1(t)]$; $y_2(t) = k\Delta x_2(t)$

$y(t) = [x_1(t+1) - x_1(t) + x_2(t+1) - x_2(t)].k$

S_2: $y(t) = k[x_1(t+1) - x_1(t) + x_2(t+1) - x_2(t)]$

$S_1 = S_2$ (Linear)

Time Variant And Time Invariant Systems

→ A system is said to be time invariant if the system does not depend on time i.e system delay is not function of time.

Ex:

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

$$x(t) \longrightarrow y(t)$$

$$x(t-t_0) \longrightarrow y(t-t_0)$$

→ A time shift t_0 in the input results in the same amount of time shift in the o/p but the waveshape does not change.

i.e the i/p and/ to o/p characteristics does not change with time.

→ Any system which doesnot obey the above principle is called as time varying system.

→ An electrical system is said to be time invariant if its component values (R, L, C) does not change with time.

Check for time Invariant:

1. Shift the i/p only and get the o/p.
2. Shift the entire system and get the o/p.
3. If both steps are identical for o/p then it is time invariant system.

Ex!

(1) $y(t) = 4x(t)$

Sol $S_1: y(t) = 4x(t-1)$
$S_2: y(t-1) = 4x(t-1)$ $\rightarrow S_1 = S_2$ (TIV)

(2) $y(t) = 4t\,x(t)$

Sol $S_1: y(t) = 4t\,x(t-1)$
$S_2: y(t-1) = 4(t-1)(x(t-1))$ $\rightarrow S_1 \neq S_2$ (TV)

(3) $y(t) = a\,x(t)$

Sol $S_1: y(t) = a\,x(t-1)$
$S_2\ y(t-1) = a\,x(t-1)$ $\rightarrow S_1 = S_2$ (TIV)

(4) $y(t) = a\,x(t) + b$

Sol $S_1: y(t) = a\,x(t-1) + b$
$S_2: y(t-1) = a\,x(t-1) + b$ $S_1 = S_2$ (TIV)

(5) $y(t) = 5t\,[x(t)]^2$

Sol $S_1: y(t) = 5t\,[x(t-1)]^2$
$S_2: y(t-1) = 5(t-1)[x(t-1)]^2$ $S_1 \neq S_2$ (TV)

(6) $y(t) = x(t+1)e^{-t}$

Sol $S_1: y(t) = x(t+1-1)e^{-t} = x(t)e^{-t}$
$S_2: y(t-1) = x(t+1-1)e^{-(t-1)}$
$= x(t) \cdot e^{-t} e^{1}$ → constant
$S_1 = S_2$ (TIV)

⑦ $y(t) = x(t+3)$

Sol S1: $y(t) = x(t+3-1)$

S2: $y(t-1) = x(t+3-1)$ $\Big] \rightarrow S_1 = S_2$ (TIV)

$= x(t+2)$

⑧ $y(t) = x^2(t)$

Sol S1: $y(t) = x^2(t-1)$

S2: $y(t-1) = x^2(t-1)$

$\therefore S_1 = S_2$ (TIV)

⑨ $y(t) = e^{x(t)}$

Sol S1: $y(t) = e^{x(t-1)}$

S2: $y(t-1) = e^{x(t-1)}$

$S_1 = S_2$ (TIV)

Linear Time Invariant System (LTI):

→ Any system which obeys the linearity and time invariant property is called as LTI system.

Linear Time Variant System (LTV):

→ Any system which obeys the linearity and does not obey time invariant property is called LTV system.

Ex: $y(t) = a\,x(t)$

Linearity: $y_1(t) = a x_1(t)$; $y_2(t) = a x_2(t)$

$y(t) = a x_1(t) + a x_2(t)$

$y(t) = a[x_1(t) + x_2(t)]$

S2: $y(t) = a[x_1(t) + x_2(t)]$.

$\therefore S_1 = S_2$.

T.I:

$y(t) = a\,x(t)$

S1: $y(t) = a\,x(t-1)$

S2: $y(t-1) = a\,x(t-1)$ $\Big] \rightarrow S_1 = S_2$ (TIV)

$\therefore$ It is a linear time invariant system (LTI)

Ily:

(2) $y(t) = t\,x(t) \rightarrow$ LTI

(3) $y(t) = a\,x(t) + b \rightarrow$ NLTI

(4) $y(t) = a x^2(t) \rightarrow$ NLTI

(5) $y(t) = e^{x(t)} \rightarrow$ NLTI

(6) $y(t) = x(t - t_0) \rightarrow$ LTI

Stable System:

→ System is absolutely integrable

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$$

Causal And Non Causal Systems:

→ A system is said to be causal if o/p $y(t_0)$ depends only on the values of i/p $x(t)$ at $t < t_0$ {present, i/p, past i/ps, past o/ps}

$x(t) = 0$, for $t < 0$

non causal $t \leq 0$, or $t \leq 0$ and $t > 0$

$x(t) \neq 0$ for $t < 0$

Ex: $y(t) = 4x(t-1)$

$y(2) = 4x(2-1) \Rightarrow 4x(2)$

$y(t) = 4x(t-1) + x(t)$

$y(2) = 4x(1) + x(2)$

→ A system is said to be non-causal if the o/p depends on future values of i/p i.e future i/ps & o/ps.

Ex: $y(t) = 4x(t+1)$

$y(2) = 4x(3)$

Examples whether it is causal & Non causal:

(1) $y(t) = K[x(t+1) - x(t)]$

$y(0) = K[x(1) - x(0)]$ → Noncausal

(2) $y(t) = 3x(t+3)$

$y(0) = 3x(3)$ → Non causal

(3) $y(t) = (t+3)x(t-3)$

$y(0) = (0+3)x(0-3)$

$= 3x(-3)$ → causal

(6) $y(t) = x(2t)$ → Non causal

(7) $y(t) = x(t) - x(t-1)$ → causal

(8) $y(t) = x(t) + \int_0^t x(\lambda)\, d\lambda$

$= x(t) + z(\lambda)]_0^t \Rightarrow$ Causal.

At $t=0, t=1, t=2$

(4) $y(t) = x(t) + 3x(t+4)$

when $t=0$, $y(0) = x(0) + 3x(4)$

when $t=1$, $y(1) = x(1) + 3x(5)$

So here response at $t=0$, $y(0)$ depends on the present i/p & future i/p

hence system is noncausal.

(5) $y(t) = x(t^2)$

$t=-1$, $y(-1) = x(1)$ → future

$t=0$, $y(0) = x(0)$ → present

$t=1$, $y(1) = x(1)$ → present

$t=2$, $y(2) = x(4)$ → future

Non causal.

Except at $t=0$, $t=1$, the response of any value of t depends on future i/p.

Impulse Response:

The response of a system for an impulse i/p is called a impulse response of the system and it is denoted by h(t)

$$\delta(t) \longrightarrow \boxed{\text{system}} \longrightarrow h(t)$$

$$\delta(t) \rightarrow h(t)$$

→ Every system is characterised by its impulse response.

$$\underset{f(t)}{i/p} \longrightarrow \boxed{h(t)} \longrightarrow \underset{r(t)}{o/p}$$

Response of a System for an arbitary i/p:

Response of $\delta(t) \rightarrow h(t)$

$$\delta(t-t_0) \rightarrow h(t-t_0)$$

$$\delta(t) + \delta(t-t_0) = h(t) + h(t-t_0)$$

The response of a system for a given i/p f(t) is determined by using superposition principle.

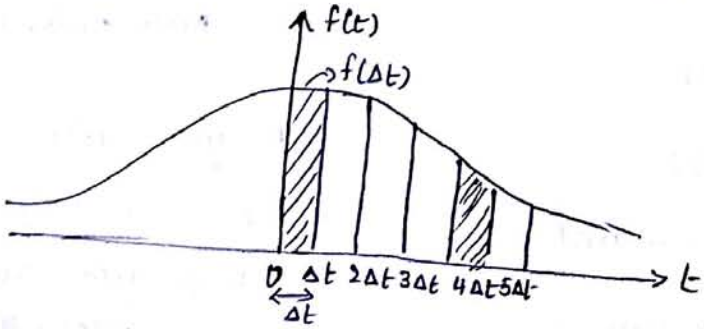
step 1: Resolve the i/p function interms of impulse functions.

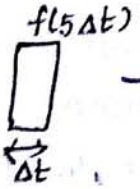
step 2: Determine individually the response of LTI system for impulse function.

step 3: Find the sum of individual responses which will become the overall response r(t).

Representation of a function f(t) in terms of an impulse function:

Here the function f(t) is a impulse train function.





→ area $f(5\Delta t) \times \Delta t$

$$f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta t)\,\Delta t \cdot \delta(t-n\Delta t)$$

$$f(t) = \underset{\Delta t \to 0}{Lt} \sum_{n=-\infty}^{\infty} f(n\Delta t)\, \Delta t\, \delta(t - n\Delta t)$$

The rectangular of width Δt & height $f(n\Delta t)$ and area under the rectangles is $\Delta t \cdot f(n\Delta t)$ and this n^{th} element approached a delta function of strength $f(n\Delta t)\,\Delta t$ located at $t = n\Delta t$. and this delta function is represented as $f(n\Delta t)\,\Delta t\, \delta(t - n\Delta t)$

$$\boxed{f(t) = \underset{\Delta t \to 0}{Lt} \sum_{n=-\infty}^{\infty} f(n\Delta t) \cdot \Delta t \cdot \delta(t - n\Delta t)}$$

As $\Delta t \to 0$, the nth element may be considered.

2) Determination of $r(t)$ for the input $f(t)$:

Let $h(t)$ be the impulse response of the system.

$$\delta(t) \longrightarrow \boxed{\text{system}} \longrightarrow h(t)$$

then $\delta(t) \to h(t)$

$$\delta(t - n\Delta t) \longrightarrow h(t - n\Delta t)$$

$$f(n\Delta t)\, \delta(t - n\Delta t) \longrightarrow f(n\Delta t) \cdot h(t - n\Delta t)$$

$$f(n\Delta t) \cdot \Delta t\, \delta(t - n\Delta t) \longrightarrow f(n\Delta t) \cdot \Delta t\, h(t - n\Delta t)$$

$$\underset{\Delta t \to 0}{Lt} \sum_{n=-\infty}^{\infty} f(n\Delta t) \cdot \Delta t\, \delta(t - n\Delta t) \longrightarrow \underset{\Delta t \to 0}{Lt} \sum_{n=-\infty}^{\infty} f(n\Delta t) \cdot \Delta t\, h(t - n\Delta t)$$

$$f(t) \longrightarrow \boxed{\text{system}} \longrightarrow r(t)$$

$$r(t) = \underset{\Delta t \to 0}{Lt} \sum_{n=-\infty}^{\infty} f(n\Delta t) \cdot \Delta t\, h(t - n\Delta t)$$

$\Delta t \to 0$ means summation becomes integration.

$$r(t) = \int_{-\infty}^{\infty} f(\tau) \cdot h(t - \tau)\, d\tau$$

$$r(t) = f(t) \otimes h(t)$$

$$f(t) \longrightarrow \boxed{h(t)} \longrightarrow r(t) \rightarrow f(t) \otimes h(t).$$

i.e if the response of a system is known for unit impulse fn then response to any other function f(t) can be obtained from the above eqn.

→ An unit impulse function is called as a Test function and it is used to characterise a system.

$$r(t) = f(t) \otimes h(t)$$

In frequency domain $r(t) \xleftrightarrow{FT} R(\omega)$

$f(t) \xleftrightarrow{FT} F(\omega)$

$h(t) \xleftrightarrow{FT} H(\omega)$

Using convolution property

$$f(t) \otimes h(t) = F(\omega) \cdot H(\omega)$$

$$R(\omega) = F(\omega) \cdot H(\omega)$$

$$\boxed{H(\omega) = \frac{R(\omega)}{F(\omega)}}$$

→ When $F(\omega) = 1$; i.e i/p is unit impulse $H(\omega) = R(\omega)$

→ Transfer function $H(\omega)$ of a system is defined as the transform of the response of a system where the i/p is unit impulse function.

$$H(\omega) = |H(\omega)| e^{j\theta(\omega)}$$

$\theta(\omega)$ → phase response of the system.

$|H(\omega)|$ → Amplitude response of the system.

$$\boxed{\begin{aligned} \ln[H(\omega)] &= \ln[|H(\omega)|] + j\theta(\omega) \\ &= L(\omega) + j\theta(\omega) \end{aligned}}$$

$L(\omega)$ → Gain of the system

$\theta(\omega)$ → phase shift introduced by system.

Note: An impulse function contains all frequencies in equal amount so we can use it as a test function.

$$H(\omega) = \frac{R(\omega)}{F(\omega)}$$ → Transfer fn of LTI system.

$FIT[h(t)]$

$$h(t) = I \cdot FIT[H(\omega)]$$

FILTER CHARACTERISTICS OF LINEAR SYSTEMS :

IDEAL LOW PASS FILTERS :

→ It transmits all the signals below certain frequency 'B' Hz without any distortion.

→ The range of frequencies from 0Hz to 'B' Hz is called passband of lowpass filter.

→ The frequency 'B' Hz is called cut-off frequency of the ideal lowpass filter.

→ The transfer function of ideal low pass filter can be written as

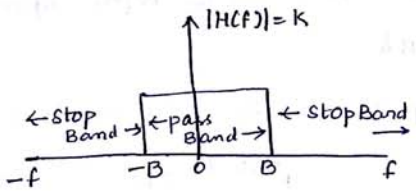
$$H(f) = K e^{-j2\pi f t_0} \quad ; \; -B \le f \le B$$
$$= 0 \quad ; \; |f| > B.$$

K = amplitude is assumed to be unity.

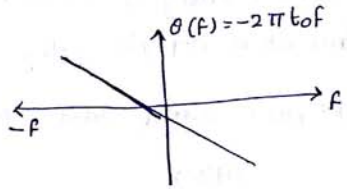
→ By K=1 in above eqn

$$H(f) = e^{-j2\pi f t_0} \quad ; \; -B \le f < B$$
$$= 0 \quad ; \; |f| > B$$

→ By inverse fourier transform, h(t) can be obtained for ideal LPF



a) Magnitude response.



(b) phase response

$$h(t) = \int_{-B}^{B} e^{-j2\pi f t_0} \cdot e^{j2\pi f t} \, df$$

$$= \int_{-B}^{B} \left[e^{j2\pi f (t-t_0)} \right] df = \frac{1}{j2\pi(t-t_0)} \left[e^{j2\pi f(t-t_0)} \right]_{-B}^{B}$$

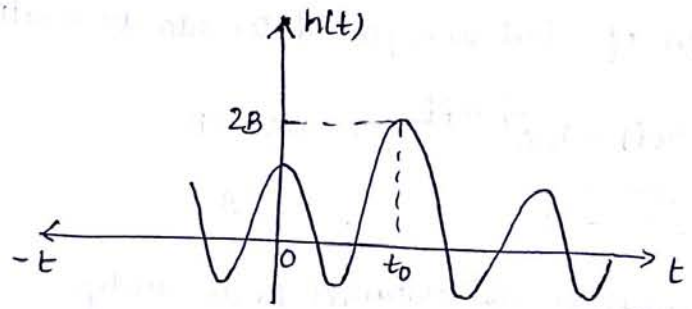
$$= \frac{1}{j2\pi(t-t_0)} \left[e^{j2\pi B(t-t_0)} - e^{-j2\pi B(t-t_0)} \right]$$

$$= \frac{1}{\pi(t-t_0)}\left[\frac{e^{j2\pi B(t-t_0)} - e^{-j2\pi B(t-t_0)}}{2j}\right]$$

$$= \frac{1}{\pi(t-t_0)} \sin\left[2\pi B(t-t_0)\right]$$

$$h(t) = 2B\left(\frac{\sin[2\pi B(t-t_0)]}{2\pi B(t-t_0)}\right) = 2B\,\mathrm{sinc}[2B(t-t_0)]$$

Response



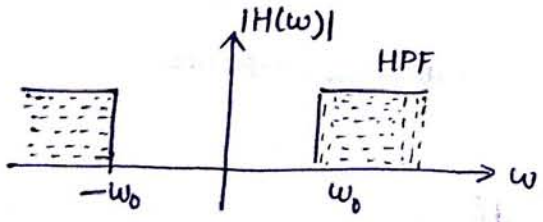
→ Figure shows that impulse response exists for negative values of 't'. But actually unit impulse is applied at $t=0$ always.

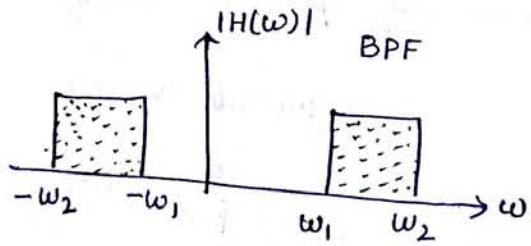
→ Practically it is impossible to implement such a system.

OTHER IDEAL FILTERS SUCH AS HPF, BPF etc.,

→ In realizability of ideal LPF its response begins before input is applied and hence it is not physically realizable.

→ $||^{ly}$ HPF, BPF ideal have frequency response as shown in figure

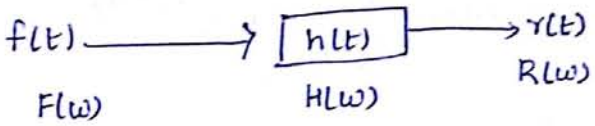




→ These have sharp transition in frequency response.

→ All ideal filters are physically not realizable since their impulse response is non-causal.

INTRODUCTION FOR FILTER CHARACTERISTICS:



$$R(\omega) = F(\omega) \cdot H(\omega)$$

→ The spectrum of o/p is $F(\omega) \cdot H(\omega)$ i.e the system acts as a kind of filter to various frequency components.

→ Some frequency components are boosted in strength and some are attenuated and some remain unaffected.

→ II^{ly} each freq. component undergoes a different amount of phase shift i.e the modification is carried out according to $H(\omega)$.
↳ acts as waiting fn for two different frequencies.

DISTORTIONLESS TRANSMISSION THROUGH SYSTEM:

→ It means output signal is an exact replica of the i/p signal.

→ The difference between i/p and o/p of such system is that

1. Amplitude of the o/p signal may increase or decrease by some factor w.r. to i/p.
2. The o/p sgl may be delayed in time w.r. to i/p sgl because of system delay.

→ o/p sgl $y(t)$ can be written in terms of i/p $x(t)$ as

$$y(t) = k\,x(t - t_0)$$

k → constant, Represents change in amplitude

t_0 → time delay in transmission of signal through a system.

By taking fourier transform

$$Y(f) = F[y(t)] = F\{k\,x(t-t_0)\}$$

From time shifting property of FT

$$Y(f) = k\,X(f)\,e^{-j2\pi f t_0}$$

Transfer fn $H(f) = \frac{Y(f)}{X(f)}$

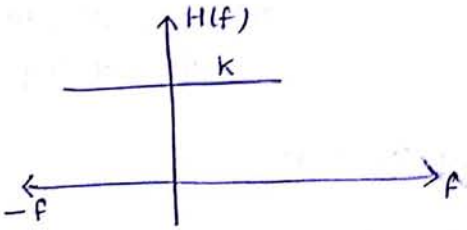
$$\therefore \quad H(f) = \frac{Y(f)}{X(f)} = k \cdot e^{-j2\pi f t_0}.$$

Magnitude of transfer fn independent of frequency.

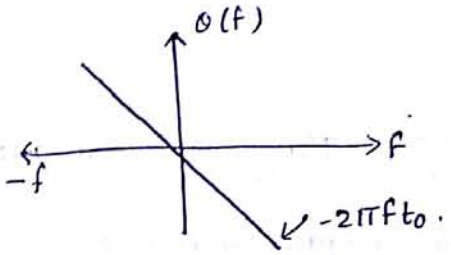
→ Transfer function has constant amplitude at all frequencies. The phase shift is

$$\theta(f) = -2\pi f t_0$$
$$= (-2\pi t_0) f.$$

→ phase shift is linearly proportional to frequency.



(a) Amplitude Spectrum



(b) phase spectrum passing through origin

→ By considering simple example

Let there be signal in time domain as

$$x(t) = \cos(2\pi f t)$$

Let the o/p sgl be same in amplitude but shifted in time by t_0 sec

$$y(t) = \cos[2\pi f (t - t_0)]$$

$$\therefore \quad y(t) = \cos(2\pi f t - 2\pi f t_0) = \cos(2\pi f t - \theta(f))$$

∴ phase shift of $y(t)$ is

$$\theta(f) = -2\pi f t_0.$$

which is proportional to frequency 'f'.

Two types :

(i) Amplitude distortion

(ii) phase distortion.

AMPLITUDE DISTORTION

→ This distortion occurs when $|H(\omega)|$ is not constant over frequency band of interest and the frequency components present in I/P sgl are transmitted with different gain and attenuation.

PHASE DISTORTION:

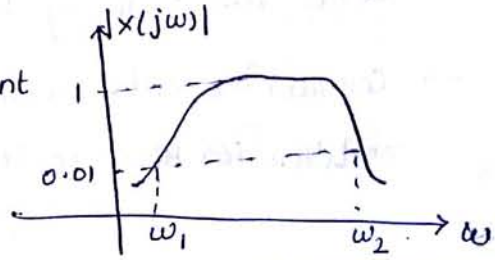
→ This distortion occurs when phase of $H(\omega)$ is not linearly changing with time and different frequency components in I/P are subjected to different time delays during transmission.

SIGNAL BANDWIDTH:

The band of frequencies that contains most of signal energy is called B.W of signal denoted by fm.

→ It is the range of significant signal frequencies which are present in the signal.

→ observe in the waveform $x(t)$ has significant frequencies from ω_1 to ω_2.

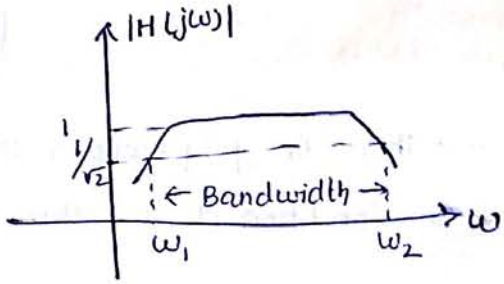


→ The B.W of this signal is $\omega_2 - \omega_1$

→ All the physically obtained signals have limited bandwidth.

SYSTEM BANDWIDTH

→ The B.W of a system is defined as range of frequencies over which $|H(\omega)|$ remains with in $1/\sqrt{2}$ times of its mid-band value. for distortionless transmission the system must have infinite B.W but physical system are limited to finite BW.



→ So a system with finite BW can provide distortionless transmission for a band limited signal if $|H(\omega)|$ remains constant over BW of the signal.

→ The range of frequencies for which magnitude $|H(j\Omega)|$ of the systems remains within $\frac{1}{\sqrt{2}}$ of its maximum value

CASUAL SYSTEMS:

→ A system is said to be causal if $h(t) = 0$ if $t < 0$

$h(t - t_0) = 0$; $t < t_0$

i.e if i/p is zero for $t < t_0$, then o/p is also zero for $t < t_0$.

→ Any system which does not obey the above rule is non-causal system.

→ If two i/p to a causal system are equal upto some time 't_0' then corresponding o/p must be equal upto that time instant.

POLY-WIENER CRITERION

→ This gives the condition for causality in frequency domain or in other words the frequency domain equivalent of causal system i.e $H(\omega)$.

→ Consider a system with transfer function $H(\omega)$, the necessary and sufficient condition for $H(\omega)$ to be transfer function of causal fn is

$$\int_{-\infty}^{\infty} \frac{|\ln(H(j\omega))|}{1+\omega^2} d\omega < \infty \rightarrow (1)$$

provided $|H(j\omega)$ is square integral.

$$\int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega < \infty \rightarrow (2)$$

→ This is poly-wiener criteria. If condition (2) is not satisfied then the condition (1) is neither necessary nor sufficient.

PHYSICAL REALIZABILITY:

→ A system is said to be physically realizable if it obeys the causal condition.

i.e $h(t) = 0$ for $t < 0$.

Ex: $H(\omega) = \frac{1}{1+j\omega}$

$h(t) = e^{-t} u(t)$

$= 0$ for $t < 0$

So the above system for transfer fn is realizable in freq. domain.

$$\int_{-\infty}^{\infty} \frac{|\ln|H(\omega)||}{H(\omega)^2} d\omega < \infty$$

→ The frequency domain statements can be interpreted as $|H(\omega)|$ of a physically realizable system may be zero for some discrete frequency but it can never be zero for a finite band of frequencies.

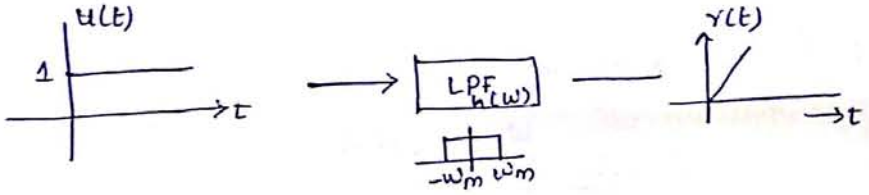
→ $H(\omega)$ for a realisable system cannot decay faster than a function of exponential order.

Ex: A system with T.F $e^{-\omega}$ is realisable where as $e^{-\omega^2}$ is not as it decay faster

RELATIONSHIP BETWEEN RISE TIME AND BANDWIDTH:

→ If a unit step fn $u(t)$ is applied to an ideal LPF, the o/p will show a gradual rise instead of a sharp rise in the i/p.

→ The rise time (t_r) is the time required by the response to reach its final value from initial value.



Transfer function of ideal low pass filter is

$$H(\omega) = |H(\omega)|\, e^{j\theta(\omega)}$$

$$= G(\omega)\, e^{-j\omega t_0}$$

↓

Rectangular pulse with magnitude K.

for $-B < f \leq B$ i.e $-\omega_m \leq \omega \leq \omega_m$ where $\omega_m = 2\pi B$.

and $\theta(\omega) = -2\pi f t_0 = -\omega t_0$.

→ Fourier transform of unit step fn $u(t)$

$$FT\{u(t)\} \Rightarrow U(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$$

→ Fourier transform of response $R(\omega)$, input and $H(\omega)$ related as

$$R(\omega) = \left[\pi\delta(\omega) + \frac{1}{j\omega}\right] H(\omega) = \pi\delta(\omega).H(\omega) + \frac{1}{j\omega} H(\omega).$$

$\delta(\omega)$ exists only for $\omega=0$ and $H(\omega)|_{\omega=0}=1$

$$R(\omega)=\pi\delta(\omega)+\frac{1}{j\omega}H(\omega).$$

By taking IFT for above eqn

$$r(t)=IFT[R(\omega)]=IFT\left\{\pi\delta(\omega)+\frac{1}{j\omega}H(\omega)\right\}$$

$$=IFT\left\{\pi\delta(\omega)+\frac{1}{j\omega}G(\omega)e^{-j\omega t_0}\right\} \quad (\because H(\omega)=G(\omega)e^{-j\omega t_0})$$

Inverse fourier transform of $\pi\delta(\omega)$ is $\frac{1}{2}$.

$$\left\{\begin{array}{l}\because 1 \rightarrow 2\pi\delta(\omega)\\ 1/2 \leftarrow \pi\delta(\omega)\end{array}\right\}$$

$$r(t)=\frac{1}{2}+IFT\left\{\frac{1}{j\omega}G(\omega)e^{-j\omega t_0}\right\}$$

$$=\frac{1}{2}+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{j\omega}G(\omega)e^{-j\omega t_0}\cdot e^{j\omega t}d\omega$$

we know $G(\omega)=1$ for $-\omega_m\leq\omega\leq\omega_m$

$$=\frac{1}{2}+\frac{1}{2\pi}\int_{-\omega_m}^{\omega_m}\frac{e^{j\omega(t-t_0)}}{j\omega}d\omega$$

$$=\frac{1}{2}+\frac{1}{2\pi}\int_{-\omega_m}^{\omega_m}\frac{\cos\omega(t-t_0)+j\sin(\omega(t-t_0))}{j\omega}d\omega$$

$$=\frac{1}{2}+\frac{1}{2\pi}\int_{-\omega_m}^{\omega_m}\frac{\cos\omega(t-t_0)}{j\omega}d\omega+\frac{1}{2\pi}\int_{-\omega_m}^{\omega_m}\frac{\sin\omega(t-t_0)}{\omega}d\omega.$$

$\downarrow$ = zero for its odd term.

$\downarrow$ even

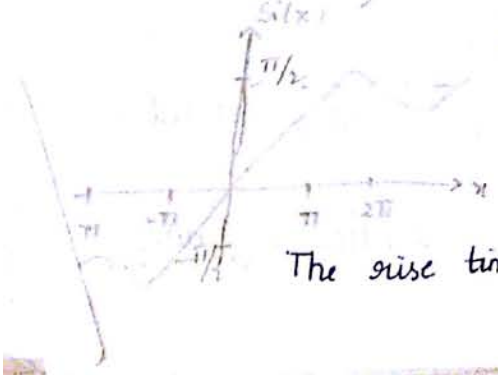
$Si(x)$ is an odd fn

$Si(-x)=-Si(x)$

(i) $Si(0)=0$

(ii) $Si(\infty)=\frac{\pi}{2}$

$Si(-\infty)=-\frac{\pi}{2}$



$$r(t)=\frac{1}{2}+\frac{1}{2\pi}\times 2\int_0^{\omega_m}\frac{\sin\omega(t-t_0)}{\omega}d\omega=\frac{1}{2}+\frac{1}{\pi}\int_0^{\omega_m}\frac{\sin\omega(t-t_0)}{\omega}d\omega$$

$$=\frac{1}{2}+\frac{1}{\pi}\left[Si\,\omega(t-t_0)\right]_0^{\omega_m}$$

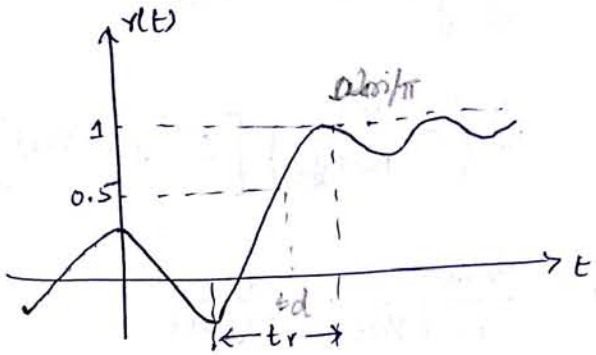
$$=\frac{1}{2}+\frac{1}{\pi}Si\,\omega_m(t-t_0) \longrightarrow \text{sine integral}$$

$$\rightarrow \frac{dr(t)}{d}\Big|_{t=t_0}=\frac{1}{\pi}\cos[\omega_m(t-t_0)].\omega_m$$

The rise time is given as $t_r=\frac{2\pi}{\omega_m}=\frac{1}{B}$

cut off frequency of LPF

$$\frac{1}{t_r}=\frac{\omega_m}{\pi}\Rightarrow t_r=\frac{\omega_m}{\pi}$$



$\omega_c \to \infty$

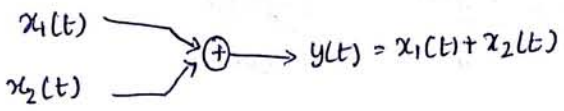
$y(t) = 1$

$\omega_c \to -\infty$

$y(t) = 0$

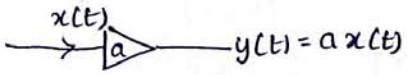
Note: {Elements of block diagram}

① Adder:



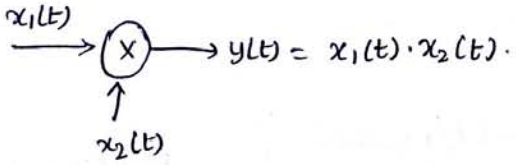
which performs the addition of two signal sequences to form sum

② constant multiplier:



It represents applying a scale factor on i/p x(t).

③ Signal multiplier:



The multiplication of two signal to form product sequence.

PROBLEMS:

① The impulse response of continuous time system is given as

$$h(t) = \frac{1}{RC} e^{-t/RC} \cdot u(t)$$

Determine the frequency response & plot the magnitude phase plots.

Sol Take FT

$$H(\omega) = \int_{-\infty}^{\infty} h(t) \cdot e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{RC} \cdot e^{-t/RC} \cdot u(t)\, e^{-j\omega t} dt$$

$$= \frac{1}{RC} \int_{0}^{\infty} e^{-t/RC} \cdot e^{-j\omega t} dt \quad \left(\because u(t) = 1 \text{ for } t \geq 0 \right.$$

step.

0 otherwise

$$= \frac{1}{RC} \int_0^\infty e^{-t(j\omega + \frac{1}{RC})}\, dt$$

$$= \frac{1}{RC}\left(-\frac{1}{j\omega + \frac{1}{RC}}\right)\left[e^{-t(j\omega + 1/RC)}\right]_0^\infty$$

$$H(\omega) = \frac{1/RC}{j\omega + 1/RC} = \frac{1}{1 + j\omega RC}.$$

Magnitude & phase

$$H(\omega) = \frac{1}{1+j\omega RC} \times \frac{1-j\omega RC}{1-j\omega RC} = \frac{1-j\omega RC}{1+(\omega RC)^2}$$

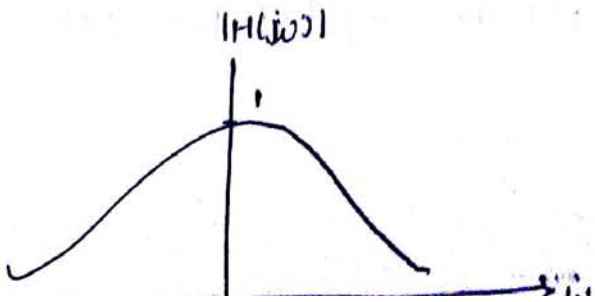
$$= \frac{1}{1+(\omega RC)^2} + j\,\frac{-\omega RC}{1+(\omega RC)^2}$$

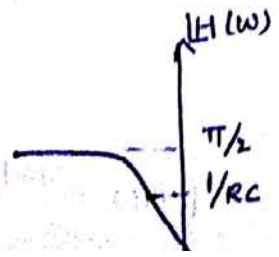
$$|H(\omega)| = \left\{\frac{1}{[1+(\omega RC)^2]^2} + \frac{(\omega RC)^2}{[1+(\omega RC)^2]^2}\right\}^{1/2}$$

$$= \frac{1}{\sqrt{1+(\omega RC)^2}}$$

$$\angle H(\omega) = \tan^{-1}\left\{\frac{(-\omega RC)/1+(\omega RC)^2}{1+[1+(\omega RC)^2]}\right\} = -\tan^{-1}(\omega RC)$$

If $RC = 1$, $|H(\omega)| = \frac{1}{\sqrt{1+\omega^2}}$; $\angle H(\omega) = -\tan^{-1}(\omega)$.





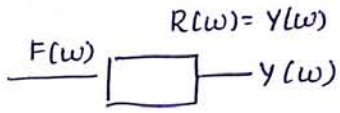
(2) For the system shown find the T.T & impulse response of the system.

$f(t) = e^{-at} \quad t>0$

$0 \quad$ elsewhere $\quad ; \quad Y(\omega) = \dfrac{1}{\alpha + j\omega}$

Sol

$$H(\omega) = \frac{R(\omega)}{F(\omega)}$$



$$F(t) = e^{-at}$$

$$F(\omega) = \frac{1}{a+j\omega} \quad ; \quad Y(\omega) = \frac{1}{\alpha + j\omega}$$

$$H(\omega) = \frac{1/\alpha+j\omega}{1/a+j\omega} = \frac{a+j\omega}{\alpha+j\omega}$$

$$F^{-1}\left[\frac{a+j\omega}{\alpha+j\omega}\right] \Rightarrow \frac{a+\alpha-\alpha+j\omega}{\alpha+j\omega} = \frac{a-\alpha}{\alpha+j\omega} + \frac{\alpha+j\omega}{\alpha+j\omega}$$

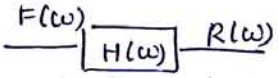
$$= \frac{a-\alpha}{\alpha+j\omega} + 1$$

$$\boxed{h(t) = (a-\alpha)\, e^{-\alpha t} u(t) + \delta(t)}$$

(3) The linear system impulse response is $[e^{-2t} + e^{-3t}]\, u(t)$ find the excitation to produce an o/p of $t \cdot e^{-2t} u(t)$?

Sol

$$h(t) = [e^{-2t} + e^{-3t}]\, u(t)$$



$$r(t) = t \cdot e^{-2t} u(t)$$

$$H(\omega) = \frac{R(\omega)}{F(\omega)}$$

$$F(\omega) = \frac{R(\omega)}{H(\omega)}$$

$$r(t) = t \cdot e^{-2t} u(t) \xleftrightarrow{F.T} \frac{1}{(2+j\omega)^2} \qquad \left(\because t \cdot e^{-at} u(t) \longleftrightarrow \frac{1}{(a+j\omega)^2} \right)$$

$$R(\omega) = \frac{1}{(2+j\omega)^2}$$

$$h(t) = e^{-2t}u(t) + e^{-3t}u(t)$$

$$H(\omega) = \frac{1}{2+j\omega} + \frac{1}{3+j\omega}$$

$$R(\omega) = \frac{1/(2+j\omega)}{\frac{3+j\omega+2+j\omega}{(2+j\omega)(3+j\omega)}} = \frac{1}{2+j\omega} \times \frac{3+j\omega}{5+2j\omega}$$

$$\frac{3+j\omega}{(2+j\omega)(5+2j\omega)} = \frac{A}{2+j\omega} + \frac{B}{5+2j\omega}$$

$$3+j\omega = A(5+2j\omega) + B(2+j\omega)$$

put $3+j\omega = 5A+2B + j\omega(2A+B)$

put $j\omega = 0$; put $j\omega(-2)$

$(3 = 5A+2B) \times 1$

$(1 = 2A+B) \times 2$

$A = 1, \; B = -1$

$$R(\omega) = \frac{1}{2+j\omega} - \frac{1}{5+2j\omega} = \frac{1}{2+j\omega} - \frac{1}{2[5/2+j\omega]}$$

$$\boxed{r(t) = e^{-2t}u(t) - \frac{1}{2}e^{-5/2\,t}u(t)}$$

DIFFERENTIAL EQUATION:

→ To obtain frequency response & impulse response.

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t).$$

differentiation property of FT is

$$\frac{d}{dt} x(t) \xleftrightarrow{FT} j\omega X(\omega).$$

$$\sum_{K=0}^{N} a_K (j\omega)^K Y(\omega) = \sum_{K=0}^{M} b_K (j\omega)^K X(j\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{K=0}^{M} b_K (j\omega)^K}{\sum_{K=0}^{N} a_K (j\omega)^K}$$

system transfer fn.

PROBLEMS:

① The differential equation of system is given as $\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = -\frac{dx(t)}{dt}$

Determine the frequency response & impulse response.

Sol

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = -\frac{dx(t)}{dt}$$

Taking F.T

$$(j\omega)^2 Y(\omega) + 5(j\omega)Y(\omega) + 6Y(\omega) = -j\omega X(\omega)$$

$$Y(\omega)\left[(j\omega)^2 + 5j\omega + 6\right] = -j\omega X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{-j\omega}{(j\omega)^2 + 5j\omega + 6}$$

$$H(\omega) = \frac{-j\omega}{(j\omega+2)(j\omega+3)} = \frac{A}{j\omega+2} + \frac{B}{j\omega+3}$$

$$= \frac{2}{j\omega+2} - \frac{3}{j\omega+3}$$

$$h(t) = \left[2 \cdot e^{-2t} - 3e^{-3t}\right] u(t)$$

impulse response of the system.

$$\left\{ \because e^{-at} u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega} \right\}$$

(2) The input voltage to the RC circuit is given by $x(t) = te^{-t/RC}u(t)$ and impulse response of this circuit is given by $h(t) = \frac{1}{RC}e^{-t/RC}u(t)$. Find output $y(t)$

Sol

output $y(t) = x(t) * h(t)$

In frequency domain

$$Y(\omega) = X(\omega)H(\omega)$$

and $H(\omega) = F[h(t)]$

$$H(\omega) = F\left\{\frac{1}{RC}e^{-t/RC}.u(t)\right\}$$

$$= \frac{1}{RC}\cdot\frac{1}{\frac{1}{RC}+j\omega} = \frac{1}{1+j\omega RC}$$

$$X(\omega) = F\left[t.e^{-t/RC}u(t)\right]$$

$$= \int_{-\infty}^{\infty} t.e^{-t/RC}e^{-j\omega t}dt = \frac{1}{\left(\frac{1}{RC}+j\omega\right)^2} = \frac{(RC)^2}{(1+j\omega RC)^2}.$$

$$\left[\because te^{-at}u(t) \xleftrightarrow{FT} \left(\frac{1}{a+j\omega}\right)^2\right]$$

$$Y(\omega) = X(\omega).H(\omega)$$

$$= \frac{(RC)^2}{(1+j\omega RC)^2}\cdot\frac{1}{(1+j\omega RC)} = \frac{(RC)^2}{(1+j\omega RC)^3}$$

$$y(t) = F^{-1}\{Y(\omega)\} = F^{-1}\left\{\frac{(RC)^2}{(1+j\omega RC)^3}\right\} = F^{-1}\left\{\frac{\cancel{(RC)^2}}{\cancel{(RC)^3}\left(\frac{1}{RC}+\right.}\right.$$

$$\boxed{y(t) = \frac{1}{RC}\cdot\frac{t^2.e^{-t/RC}}{2}u(t)}$$

1) $h(t) = e^{-5|t|}$

For stability $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

$$\therefore \int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} |e^{-5|t|}|\, dt = \int_{-\infty}^{\infty} e^{-5|t|}\, dt$$

$$= \int_{-\infty}^{0} e^{5t}\, dt + \int_{0}^{\infty} e^{-5t}\, dt = \left[\frac{e^{5t}}{5}\right]_{-\infty}^{0} + \left[\frac{e^{-5t}}{-5}\right]_{0}^{\infty}$$

$= \frac{2}{5}$ = constant / so system is stable.

2) $h(t) = e^{4t} u(t)$

$$= \int_{-\infty}^{\infty} |e^{4t} u(t)|\, dt = \int_{-\infty}^{\infty} e^{4t} u(t)\, dt$$

$$= \int_{0}^{\infty} e^{4t}\, dt = \frac{e^{\infty}}{4} - \frac{e^{0}}{4} = \infty - \frac{1}{4} = \infty$$ (Unstant.

3) $h(t) = e^{-4t} u(t)$ (stable)

4) $h(t) = t \cos t\, u(t)$ (unstable.

$$\int_{0}^{\infty} t \cos t\, dt$$

5) $h(t) = e^{-t} \sin t\, u(t)$ (stable.

$$= \int_{0}^{\infty} e^{-t} \sin t\, dt$$

→ The system produces the o/p of $y(t) = e^{-t}u(t)$ for an input of $x(t) = e^{-2t}u(t)$. Determine the impulse response and frequency response of the system.

Sol

$$y(t) = e^{-t}u(t)$$
$$x(t) = e^{-2t}u(t)$$

Consider standard fourier transform pair $e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega}$

$$Y(\omega) = \frac{1}{1+j\omega}; \quad X(\omega) = \frac{1}{2+j\omega}.$$

From equation

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1/1+j\omega}{1/2+j\omega} = \frac{2+j\omega}{1+j\omega}.$$

Multiply ~~and divide~~ the numerator & denominator by $1-j\omega$

$$H(\omega) = \frac{2+j\omega}{1+j\omega} \times \frac{1-j\omega}{1-j\omega} = \frac{(2+j\omega)(1-j\omega)}{(1)^2-(j\omega)^2} = \frac{2-2j\omega+j\omega+\omega^2}{1+\omega^2}$$

$$= \frac{2+\omega^2-j\omega}{1+\omega^2} \Rightarrow \frac{2+\omega^2}{1+\omega^2} + j\frac{-\omega}{1+\omega^2}.$$

Magnitude $|H(\omega)| = \left\{\sqrt{\left[\frac{2+\omega^2}{1+\omega^2}\right]^2 + \left[\frac{\omega^2}{1+\omega^2}\right]^2}\right]$

$$|H(\omega)| = \sqrt{\frac{4+\omega^2}{1+\omega^2}}; \quad \angle H(\omega) = \tan^{-1}\left(\frac{\frac{-\omega}{1+\omega^2}}{\frac{2+\omega^2}{1+\omega^2}}\right)$$

$$= -\tan^{-1}\left(\frac{\omega}{2+\omega^2}\right\}$$

$$\therefore H(\omega) = \frac{2+j\omega}{1+j\omega} \Rightarrow \frac{1+j\omega+1}{1+j\omega} \Rightarrow 1+\frac{1}{1+j\omega}.$$

Inverse fourier transform

$$\boxed{h(t) = IFT\{H(\omega)\} = \delta(t) + e^{-t}u(t)}$$

impulse response.

(1) The transfer function of LPF is given by

$$H(\omega) = \begin{cases} (1+K\cos\omega T)e^{-j\omega T} & ; |\omega| < 2\pi B \\ 0 & ; |\omega| > 2\pi B. \end{cases}$$

Determine the output $y(t)$ when a pulse $x(t)$ bandlimited in B is applied at the input.

Sol)

$$Y(\omega) = X(\omega)\,H(\omega)$$

$$= X(\omega)\,[1+K\cos\omega T]\,e^{-j\omega T}$$

$$= X(\omega)\,e^{-j\omega T} + K\,X(\omega)\cos\omega T\,e^{-j\omega T}$$

we know

$$x(t-T) + x(t+T) \longleftrightarrow 2X(\omega)\cos\omega T$$

$$x(t-T) \longleftrightarrow X(\omega)\,e^{-j\omega T}$$

$$y(t) = F^{-1}[Y(\omega)]$$

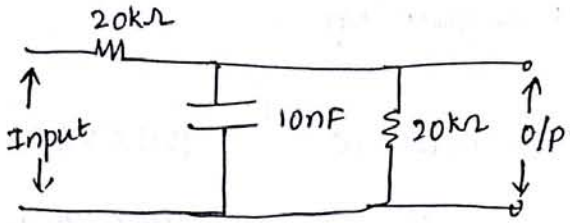
$$= F^{-1}\left[X(\omega)\,e^{-j\omega T} + K\,X(\omega)\,e^{-j\omega T}\cos\omega T\right]$$

$$= x(t-T) + \frac{K}{2}\left[x(t-T-T) + x(t-T+T)\right]$$

$$y(t) = x(t) + \frac{K}{2}\left[x(t-T) + x(t+T)\right]$$

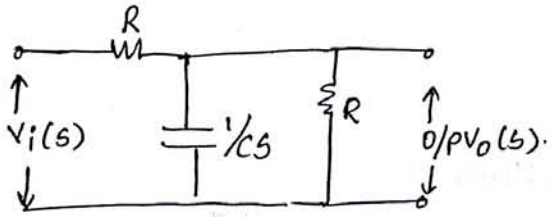
delayed by T.

2) Determine the maximum bandwidth of signals that can be transmitted through low pass RC filter as shown in figure , if over this bandwidth, the gain variation is to be 10% and the phase variation is to be within 7% of ideal characteristics.



Sol

RC network transformed into s-domain representation.



$$H(s) = \frac{V_o(s)}{V_i(s)}$$

$$= \frac{[R \parallel (1/Cs)]}{([R \parallel (1/Cs)] + R]}$$

$$= \frac{(R/Cs) / [R + 1/Cs]}{[R/Cs] / [[R + (1/Cs)] + R}$$

$$= \frac{R/(1+SCR)}{[R/(1+SCR)] + R} = \frac{R}{R + R(1+SCR)} = \frac{R}{R(1+1+SCR)}$$

$$H(s) = \frac{1}{2+SCR}.$$

But given $R = 20k\Omega$

$C = 10nF$.

$$= \frac{1}{2+(2S/10^4)}$$

$$H(s) = \frac{1}{2+S(10\times10^{-9}\times20\times10^{3})} = \frac{10^4}{2\times10^4+2S} = \frac{1}{2+S(2\times10^{-4})}$$

$$\therefore H(s) = \frac{5000}{S+10000}$$

put $s = j\omega$

$$H(\omega) = \frac{5000}{j\omega + 10000}$$

$$|H(\omega)| = \frac{5000}{\sqrt{\omega^2 + 10000}}$$

$$\phi(\omega) = -\tan^{-1}\left(\frac{\omega}{10000}\right).$$

At $\omega = 0$, $|H(\omega)|_{\omega=0} = \frac{5000}{10000} = 0.5$.

But there is 10% variation in gain over bandwidth B.

$$H(\omega) = 0.5 - 0.5 \times 10\% = 0.45$$

$$|H(\omega)| = \frac{5000}{\sqrt{B^2 + 10^8}}$$

$$B^2 + 10^8 = \left(\frac{5000}{0.45}\right)^2 \Rightarrow B^2 = 23.46 \times 10^6$$

$$B = 4.84 \text{ KHZ}$$

But $B = 2\pi f$

$$f = \frac{B}{2\pi} = \frac{4.84 \times 10^3}{2\pi} = 770.8 \text{ HZ}$$

phase at frequency, $f = 770.8$ HZ

$$\phi(\omega) = -\tan^{-1}\left(\frac{4.84}{10}\right) = -25.83°$$

(2) There are several possible ways of estimating an essential bandwidth of non-bandlimited signal. For a low pass signal, for example, the essential b.w may

UNIT-4

CONVOLUTION AND CORRELATION OF SIGNALS:

→ Convolution is used to find common area between two signals or two fns.

The convolution f(t) of two time functions $f_1(t)$ and $f_2(t)$ is designed or defined as

$$f(t) = f_1(t) \otimes f_2(t)$$

$$= \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau$$

(or)

$$= \int_{-\infty}^{\infty} f_2(\tau)\, f_1(t-\tau)\, d\tau$$

→ Convolution is a mathematical operation and multiplication is one of the form of convolution.

In convolution method

Ex:
$$\begin{array}{r} 25 \\ \times 25 \\ \hline 625 \end{array}$$

(i) $f_1(\tau)$ & $f_2(\tau)$
$\downarrow$ $\downarrow$
25 25

(ii) $f_2(-\tau)$
$\Rightarrow 52$

(iii) $f_2(t-\tau)$
$f_2(-\tau)$ shifted to right side by t seconds.

→ Here 't' is varied from $-\infty$ to ∞

Convolution Procedure:

Step 1: $f_1(t) \otimes f_2(t) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau$

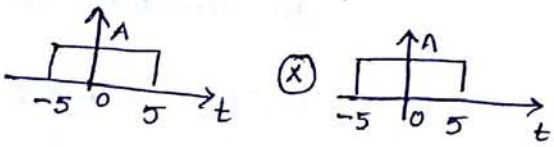
Here the independent variable convolution integral is τ, so replace t by τ to get $f_1(\tau)$ & $f_2(\tau)$

Step 2: $f_1(\tau)$ is first function and $f_2(\tau)$ is the 2nd function. $f_2(-\tau)$ is the mirror image of the $f_2(\tau)$

Step 3: $f_2(t-\tau)$ represents the function $f_2(-\tau)$ shifted to right side by t sec. 't' is varied from $-\infty$ to ∞ and find common area between two functions.

→ The value of convolution obtained at different values of 't' and may be plotted on a graph.

① Find the F.T of



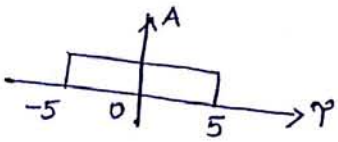
Sol Step 1:

$f_1(t) = A\,G_{10}(t) \longrightarrow f_1(\tau) = A\,G_{10}(\tau)$

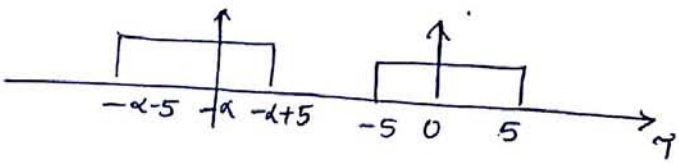
$f_2(t) = A\,G_{10}(t) \longrightarrow f_2(\tau) = A\,G_{10}(\tau)$

Step 2:

$f_2(-\tau) = A\,G_{10}(-\tau) \longrightarrow$ even function

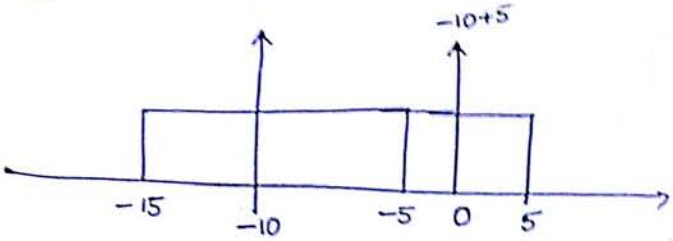


① $f_1(\tau)$ at $t = -\alpha$



$f_1(t) \otimes f_2(t) = 0$

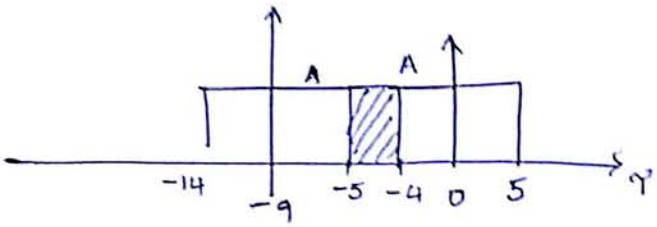
(2) at $t=-10$



$-10+5=-5$

$-10-5=-15$

(3) at $t=-9$



$-9-5=-14$

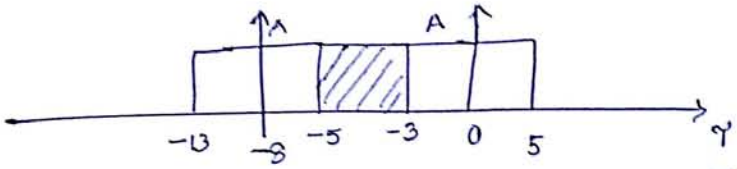
$-9+5=-4$

$$\Rightarrow \int_{-5}^{-4} A\cdot A\, d\tau = A^2[\tau]_{-5}^{-4}$$

$$= A^2[-4+5]$$

$$= A^2$$

(4) at $t=-8$



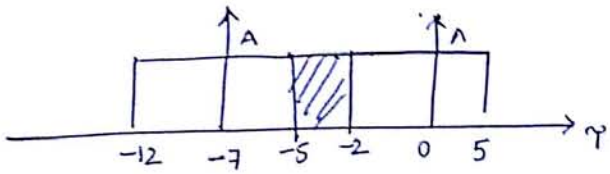
$-8+5=-3$

$-8-5=-13$

$$\Rightarrow \int_{-5}^{-3} A\cdot A\, d\tau = A^2[\tau]_{-5}^{-3}$$

$$= A^2[-3+5] \Rightarrow 2A^2.$$

(5) at $t=-7$



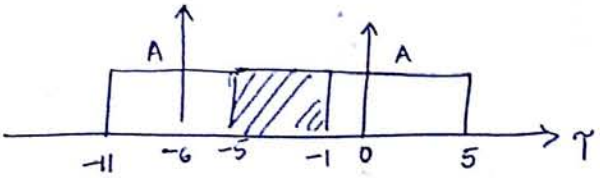
$-7+5=-2$

$-7-5=-12$

$$\Rightarrow \int_{-5}^{-2} A\cdot A\, d\tau = A^2[\tau]_{-5}^{-2}$$

$$= A^2[-2+5] = 3A^2.$$

(6) at $t=-6$



$-6+5=-1$

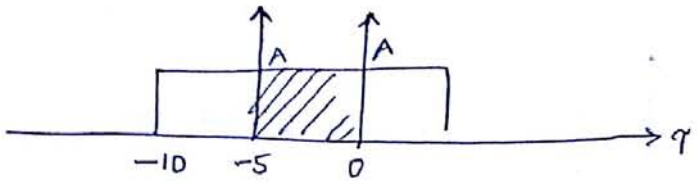
$-6-5=-11$

$$\Rightarrow \int_{-5}^{-1} A\cdot A\, d\tau \Rightarrow A^2[-1+5] = 4A^2.$$

⑦ at $t=-5$

$-5-5=-10$

$-5+5=0$

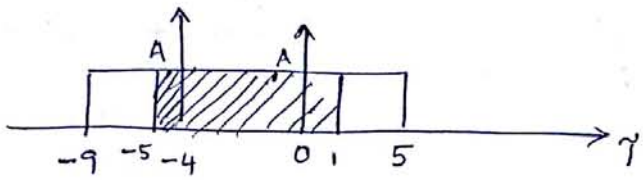


$$\int_{-5}^{0} A\cdot A\, d\tau \Rightarrow A^2\cdot[\tau]_{-5}^{0} = A^2\cdot 5$$

⑧ at $t=-4$

$-4+5=1$

$-4-5=-9$

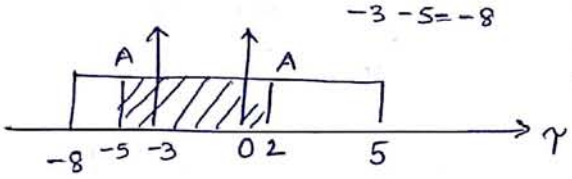


$$\int_{-5}^{1} A\cdot A\, d\tau \Rightarrow A^2[\tau]_{-5}^{1} = 6A^2.$$

⑨ at $t=-3$

$-3+5=+2$

$-3-5=-8$



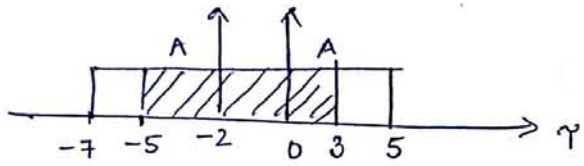
$$\int_{-5}^{2} A\cdot A\, d\tau \Rightarrow A^2[\tau]_{-5}^{2}$$

$$\Rightarrow A^2[2+5] = 7A^2.$$

⑩ at $t=-2$

$-2+5=+3$

$-2-5=-7$

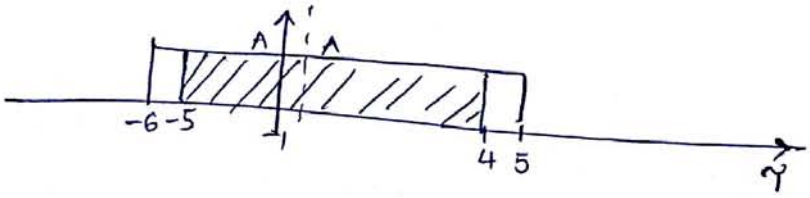


$$\int_{-5}^{3} A\cdot A\, d\tau \Rightarrow A^2[\tau]_{-5}^{3} = A^2[3+5] = 8A^2$$

⑪ at $t=-1$

$-1+5=4$

$-1-5=-6$



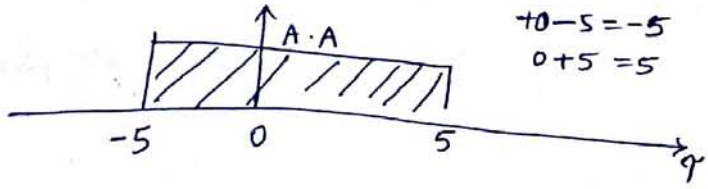
$$\Rightarrow \int_{-5}^{4} A\cdot A\, d\tau \Rightarrow A^2[\tau]_{-5}^{-4} = A^2[9]$$

$$= 9A^2$$

⑫ at $t=0$

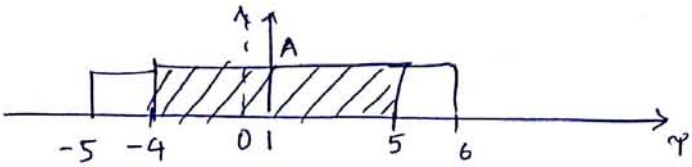
$+0-5=-5$

$0+5=5$



$$\Rightarrow \int_{-5}^{5} A\cdot A\, d\tau = A^2[\tau]_{-5}^{5} = 10A^2$$

(13) at $t=1$



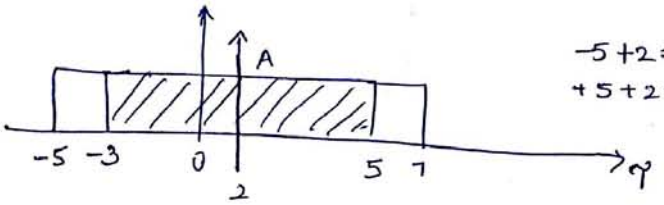
$-5+1=-4$

$+5+1=6$

$$\int_{-4}^{5} A\cdot A\, d\tau \Rightarrow A^2[\tau]_{-4}^{5}$$

$$\Rightarrow A^2[5+4]=9A^2.$$

(14) at $t=2$



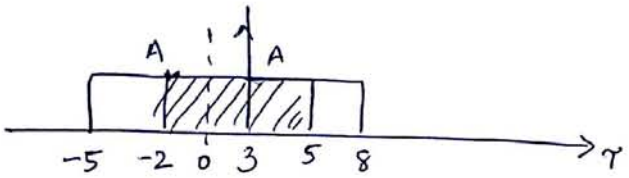
$-5+2=-3$

$+5+2=7$

$$\int_{-3}^{5} A\cdot A\, d\tau = A^2[\tau]_{-3}^{5}$$

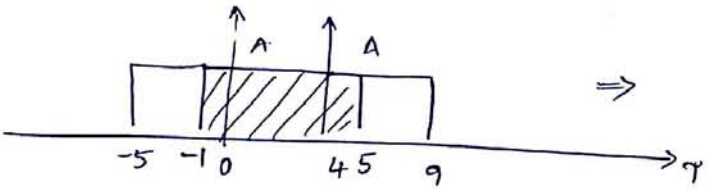
$$= 8A^2$$

(15) at $t=3$



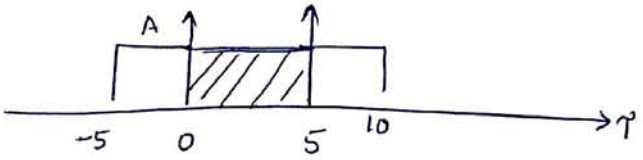
$$\int_{-2}^{5} A\cdot A\, d\tau \Rightarrow A^2[5+2]=7A^2.$$

(16) $t=4$



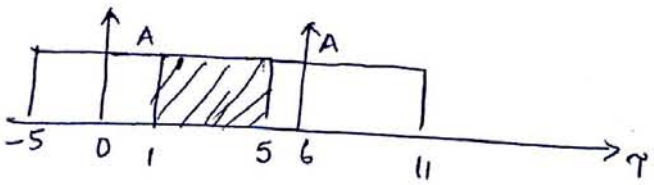
$$\Rightarrow \int_{-1}^{5} A\cdot A\, d\tau = A^2[5+1]=6A^2$$

(17) $t=5$



$$\int_{0}^{5} A\cdot A\, d\tau = A^2[5]$$

(18) $t=6$



$$\int_{1}^{5} A\cdot A\, d\tau \Rightarrow A^2[\tau]^{5} = A^2[4]$$

$\|^{ly}$ for $t=7 \Rightarrow 3A^{r}$

$t=8 \Rightarrow 2A^{r}$

$t=9 \Rightarrow 1A^{r}$

$t=10 \Rightarrow 0$

$t=-10 ; 10 = 0$

$t=-9 ; 9 = 1A^{r}$

$t=-8 ; 8 = 2A^{r}$

$t=-7 ; 7 = 3A^{r}$

$t=-6 ; 6 = 4A^{r}$

$t=-5 ; 5 = 5A^{r}$

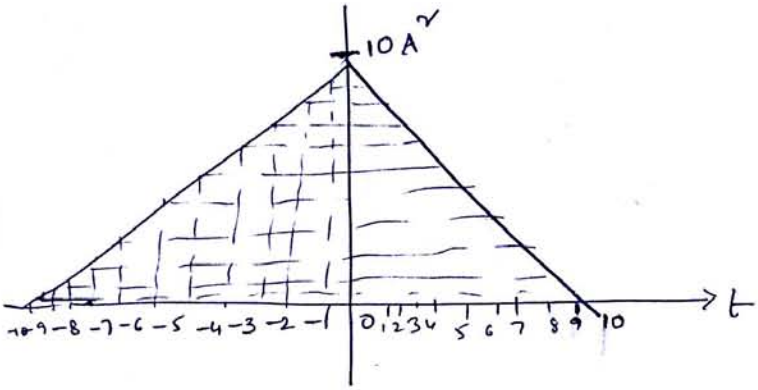
$t=-4 ; 4 = 6A^{r}$

$t=-3 ; 3 = 7A^{r}$

$t=-2 ; 2 = 8A^{r}$

$t=-1 ; 1 = 9A^{r}$

$t=0 \Rightarrow 10A^{r}$



Convolution: Mathematical way of combining two signals to form a third signal. i.e input signal, output signal and impulse response.

→ It is used to express the input and output relationship of a LTI system

→ If two functions $x(t)$ and $y(t)$ in time domain are defined then convolution $z(t)$ is

$$z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau.$$

(or)

$x(t) \otimes y(t)$.

↳ Read as $x(t)$ convolved with $y(t)$.

Convolution in time domain:

→ The convolution in time domain is equivalent to multiplication of their spectra in frequency domain i.e $x(t) \xleftrightarrow{X(\omega)}$ and $y(t) \longleftrightarrow Y(\omega)$ then

$$x(t) * y(t) \longleftrightarrow X(\omega) \cdot Y(\omega).$$

proof $x(t)$ fourier transform is given by

$$F\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$F\{x(t) * y(t)\} = \int_{-\infty}^{\infty} [x(t) * y(t)]\, e^{-j\omega t}\, dt$$

But $x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$

$$\therefore F\{x(t) * y(t)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau \right] e^{-j\omega t}\, dt$$

Interchanging the order of integration, we get

$$F\{x(t) * y(t)\} = \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} y(t-\tau)\, e^{-j\omega t}\, dt \right] d\tau$$

Using time shifting property, we get

$$\int_{-\infty}^{\infty} y(t-\tau)\, e^{-j\omega t}\, dt = Y(\omega)\, e^{-j\omega\tau}$$

$$F[x(t) * y(t)] = \int_{-\infty}^{\infty} x(\tau) Y(\omega) e^{-j\omega\tau} d\tau$$

$$= Y(\omega) \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau = Y(\omega) X(\omega).$$

$$x(t) * y(t) \longleftrightarrow X(\omega) Y(\omega)$$

This is time convolution theorem.

Convolution in frequency domain:

→ Multiplication of two functions in time domain is equivalent to convolution of their spectra in frequency domain.

$x(t) \longleftrightarrow X(\omega)$ and $y(t) \longleftrightarrow Y(\omega)$ then $2\pi x(t) y(t) \longleftrightarrow X(\omega) * Y(\omega)$ (or)

$x(t) y(t) \longleftrightarrow X(f) * Y(f)$.

proof

$$X(\omega) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = F^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$F\{x(t) y(t)\} = \int_{-\infty}^{\infty} x(t) y(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) e^{j\lambda t} d\lambda \right] y(t) e^{-j\omega t} dt$$

Interchanging the order of integration we get

$$F\{x(t) y(t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) \left[\int_{-\infty}^{\infty} y(t) \cdot e^{-j\omega t} e^{j\lambda t} dt \right] d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) \left[\int_{-\infty}^{\infty} y(t) \cdot e^{-j(\omega-\lambda)t} dt \right] d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) Y(\omega-\lambda) d\lambda. = \frac{1}{2\pi} [X(\omega) * Y(\omega)]$$

$$x(t) y(t) \longleftrightarrow \frac{1}{2\pi} X(\omega) * Y(\omega)$$

$$2\pi x(t) y(t) \longleftrightarrow X(\omega) * Y(\omega)$$

Graphical convolution:

→ The convolution by inspection provides the information needed without complicated calculations. This convolution by inspection procedure is called graphical convolution.

Procedure:

(1) $x(\tau)$ is the first function, where an independent variable 't' is replaced by dummy variable 'τ'.

(2) $y(-\tau)$ is the mirror image of $y(\tau)$ i.e $y(\tau)$ is flipped.

(3) $y(t-\tau)$ represents the function $y(-\tau)$ shifted to right by t seconds.

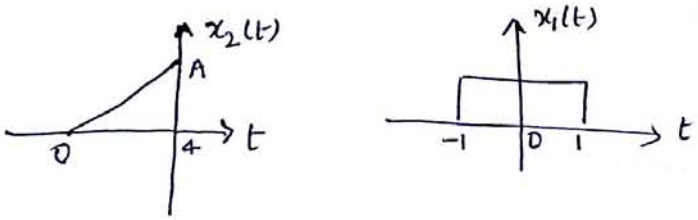
(4) For a particular value of $t=a$, integration of product $x(\tau)\,y(a-\tau)$ represents the area under the (common area) product curve.

$$\int_{-\infty}^{\infty} x(\tau)\,y(a-\tau)\,d\tau = [x(t) * y(t)]_{t=a}$$

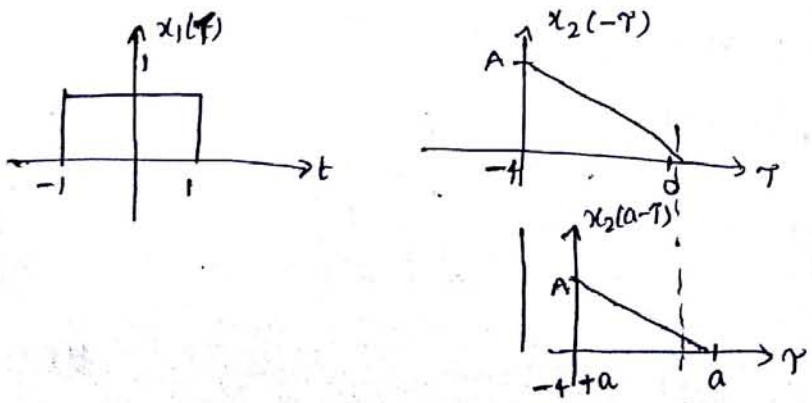
(5) The procedure is repeated for different values of t.
For negative value of t, the function $y(-\tau)$ is shifted left by t seconds.

6) The value of convolution obtained at different values of 't' (i.e +ve, -ve).

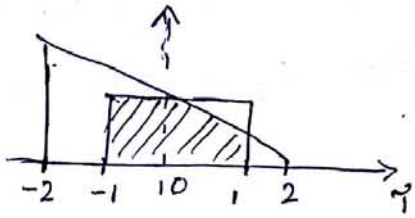
Q. Find the convolution of the functions $x_1(t)$ and $x_2(t)$



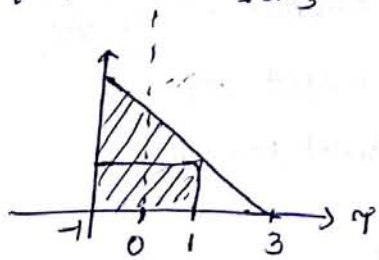
Sol)



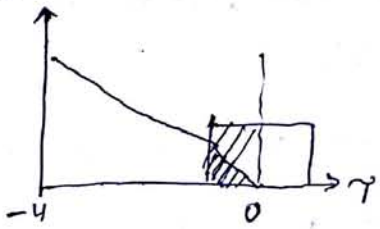
$[x_1(t) * x_2(t)]_{a=2}$

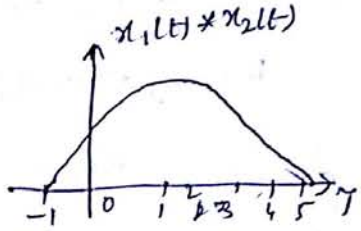


$[x_1(t) * x_2(t)]_{a=3}$



$[x_1(t) * x_2(t)]_{a=0}$



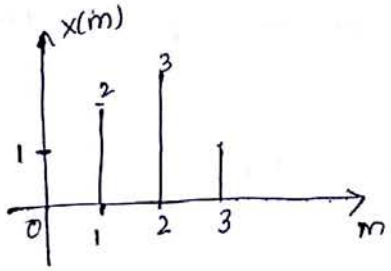


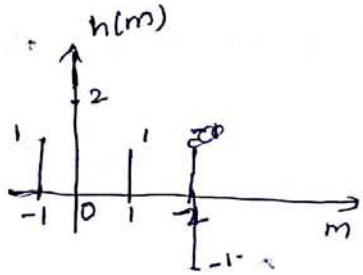
2) Determine the response of LTI system whose input $x(n)$ and impulse response $h(n)$ are given by $x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$ and $h(n) = \{\underset{\uparrow}{1}, 2, 1, -1\}$

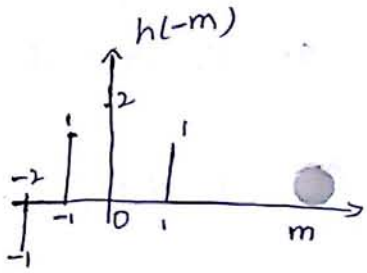
Sol The input sequence starts at $n=0$ and impulse response starts at $n=-1$.

O/p sequence starts at $n = 0+(-1) = -1$

→ i/p & impulse response consists of 4 samples, so o/p consists of $4+4-1=7$ sa

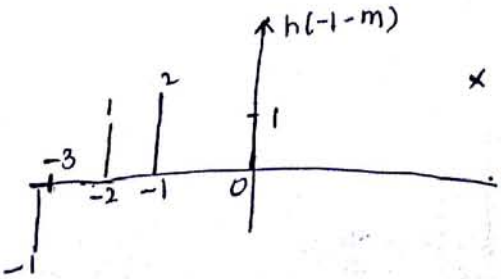






$$y(n) = \sum_{m=-\infty}^{\infty} x(m)h(n-m).$$

when $n=-1$; $y(-1) = \sum_{m=-\infty}^{\infty} x(m)\, h(-1-m)$



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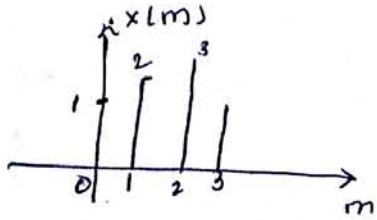
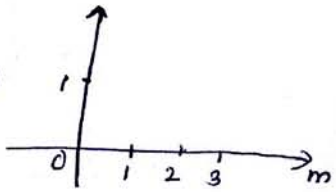


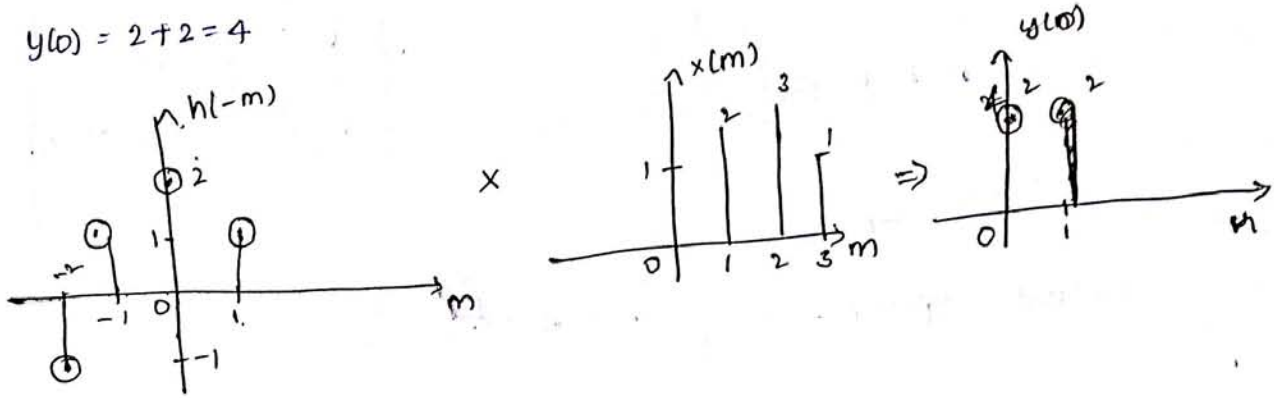
⇒



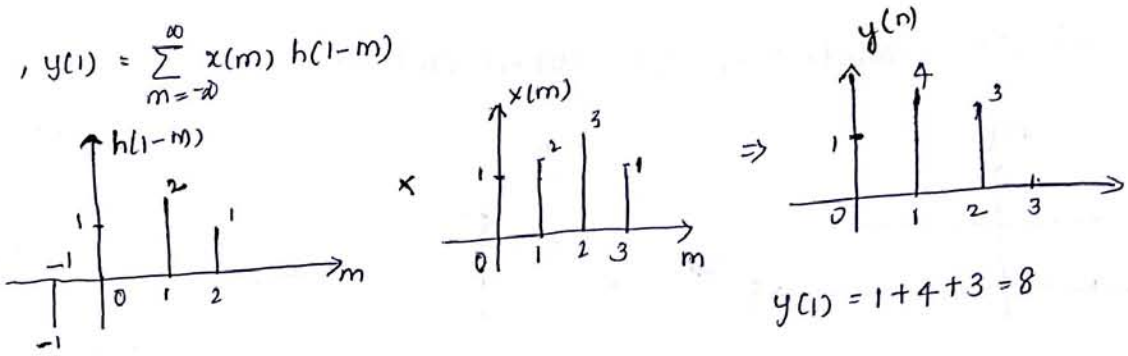
$y(-1) = 1$

when $n=0$, $y(0) = \sum_{m=-\infty}^{\infty} x(m)\, h(0-m)$.

$y(0) = 2+2 = 4$

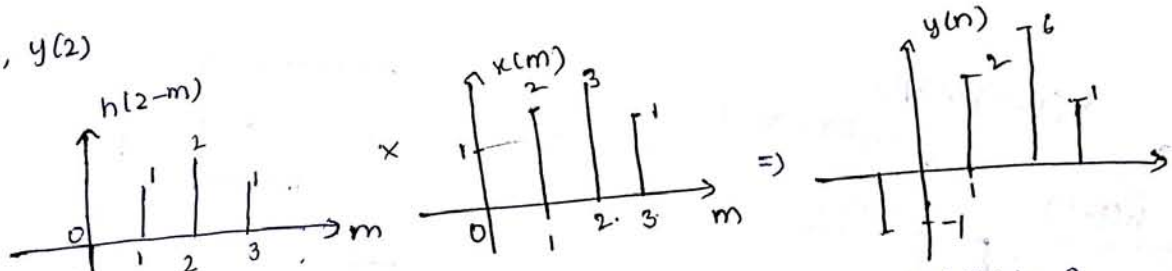


when $n=1$, $y(1) = \sum_{m=-\infty}^{\infty} x(m)\, h(1-m)$



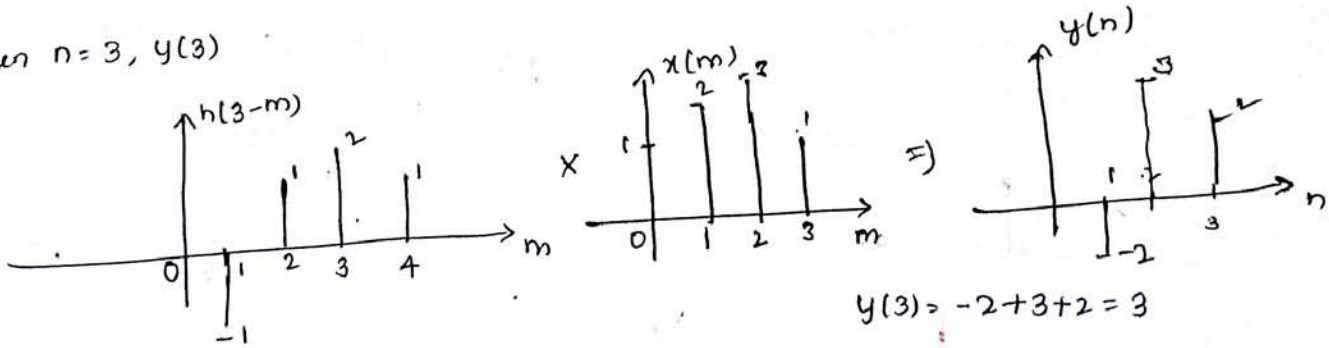
$y(1) = 1+4+3 = 8$

when $n=2$, $y(2)$



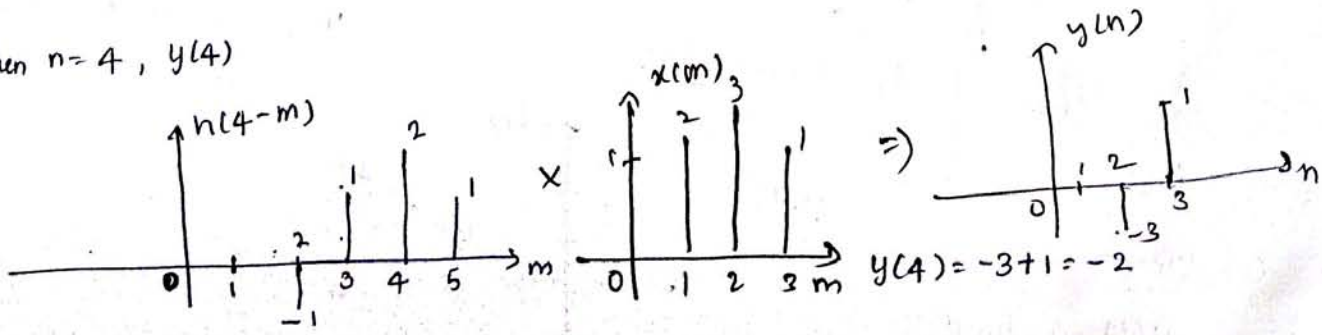
$y(2) = -1+2+6+1 = 8$

when $n=3$, $y(3)$



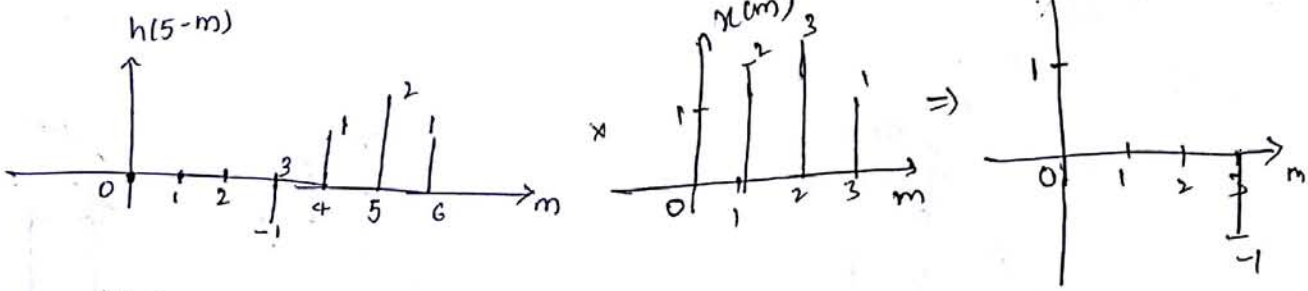
$y(3) = -2+3+2 = 3$

when $n=4$, $y(4)$



$y(4) = -3+1 = -2$

when $n = 5$

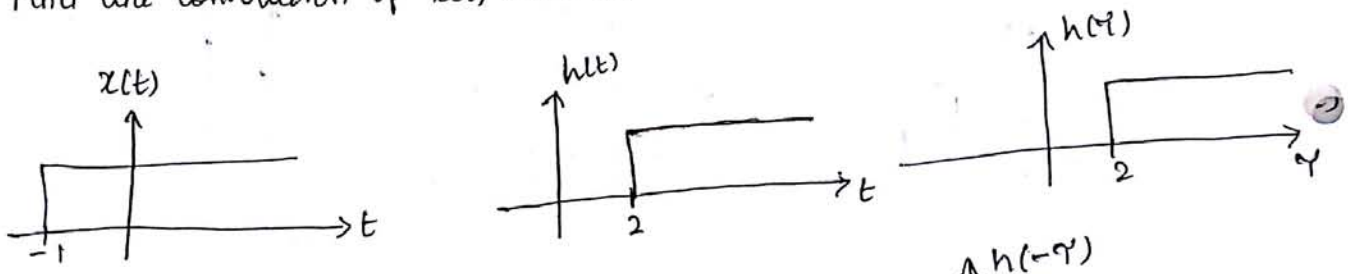


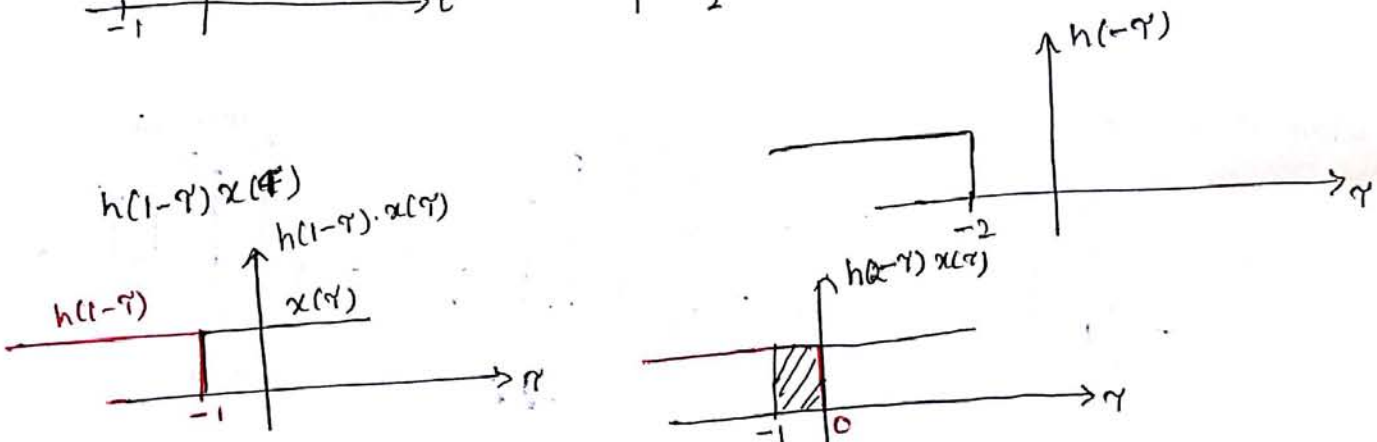
$y(5) = -1$

Output sequence $y(n) = \{1, 4, 8, 8, 3, -2, -1\}$

(3) Find the convolution of $x(t) = u(t+1)$ and $h(t) = u(t-2)$

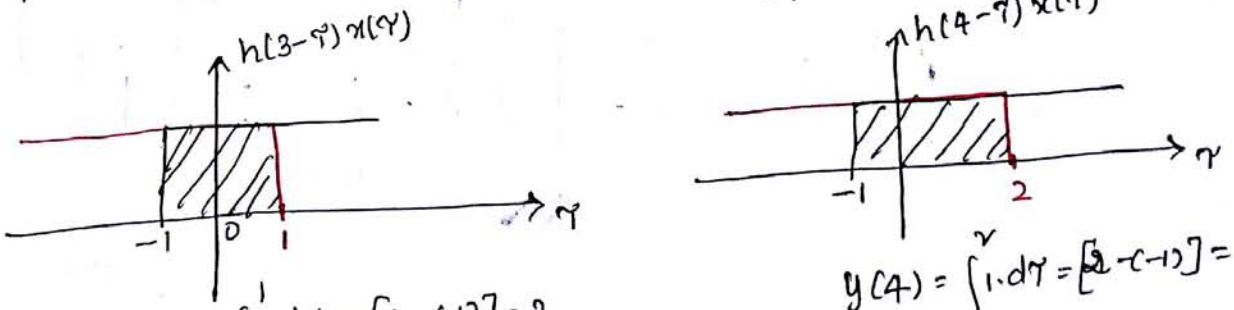
Sol





$y(1) =$ no overlap
$= 0$

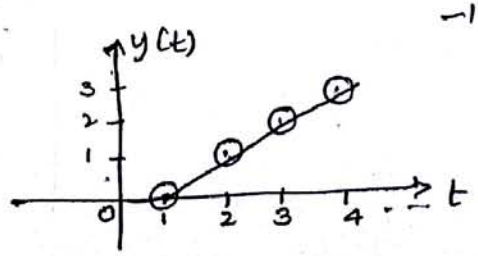
$y(2) = \int_{-1}^{0} 1 \cdot d\tau = [0-(-1)] = 1$



$y(3) = \int_{-1}^{1} 1 \cdot d\tau = [1-(-1)] = 2$

$y(4) = \int_{-1}^{2} 1 \cdot d\tau = [2-(-1)] = 3$.

$\therefore y(t) =$



Condition

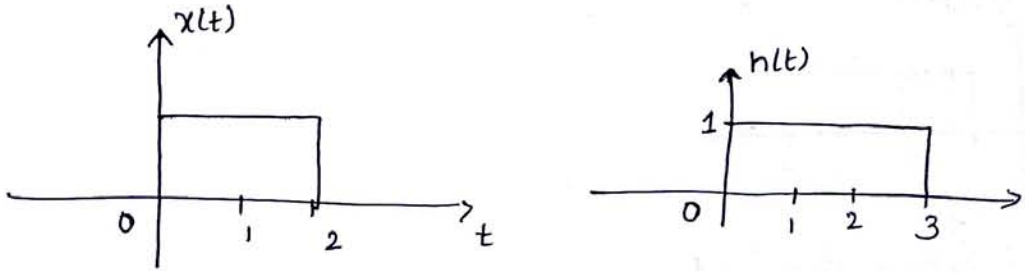
$y(t) = 0$ for $t \leq 0$

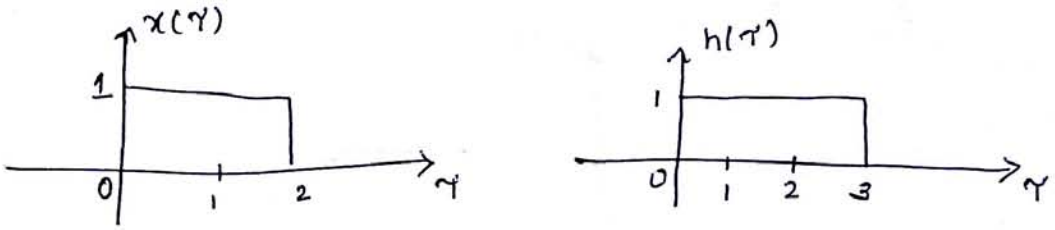
$y(t) = t-1$ for $t > 0$

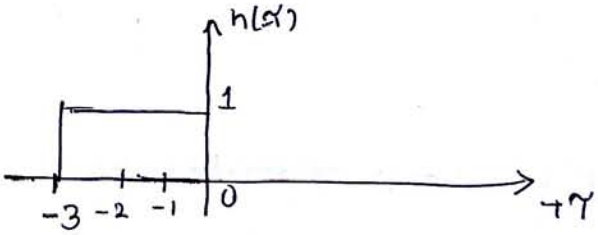
(4) Find the convolution of $x(t)$ and $h(t)$

$$x(t) = 1 \quad 0 \le t < 2$$
$$= 0 \quad \text{otherwise}$$
$$h(t) = 1 \quad 0 \le t \le 3$$
$$= 0 \quad \text{otherwise}$$

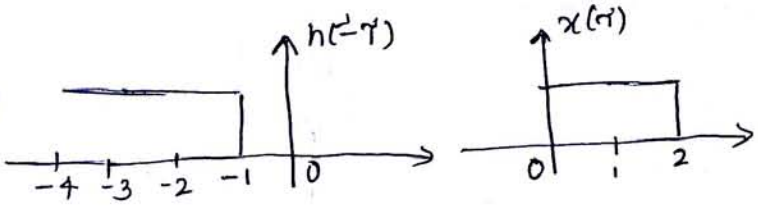
Sol





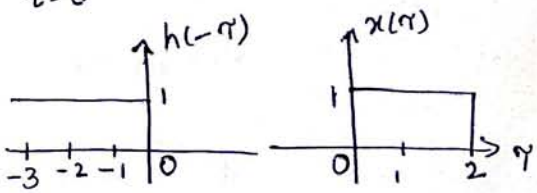


$t = -1$



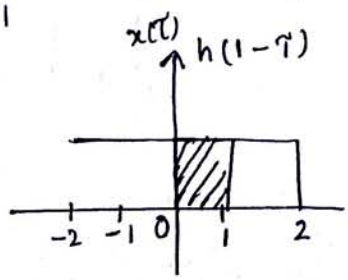
No overlap · $y(t) = 0$

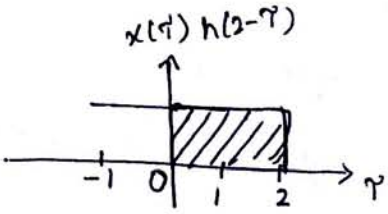
$t = 0$



$y(t) = 0$ for $t \le 0$

$t = 1$



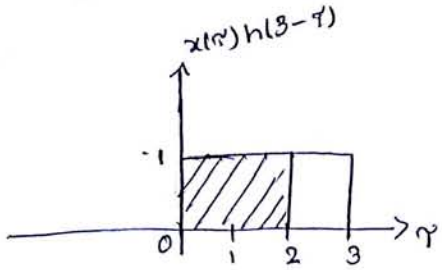


for $0 < t \le 1$

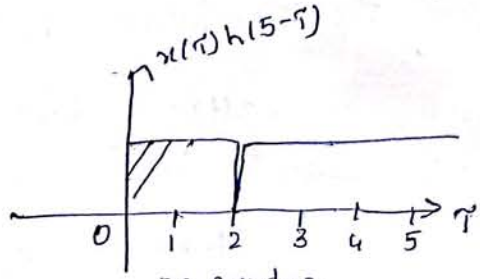
$$y(1) = \int_0^1 1 \cdot d\tau = [\tau]_0^1 = 1$$

for $0 < t \le 2$

$$y(2) = \int_0^2 1 \cdot d\tau = [\tau]_0^2 = 2.$$

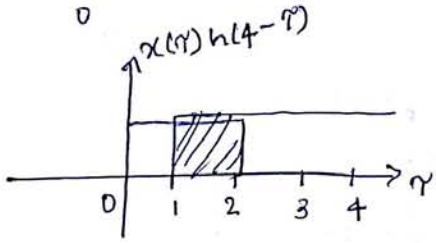


$$y(3) = \int_0^2 d\tau = 2$$

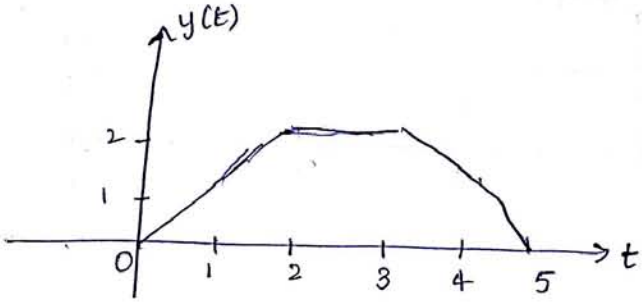


no overlap.

$y(5) = 0$.



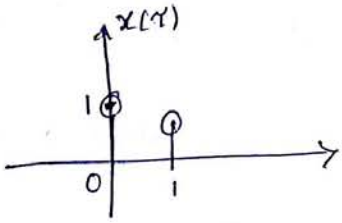
$$y(4) = \int_1^2 d\tau = [2-1] = 1$$

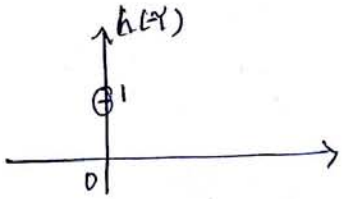


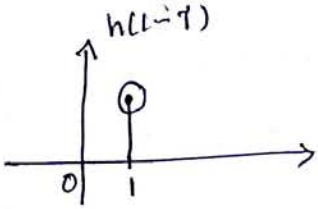
→ (5) Find the linear convolution $x(n) = \{1, 0.5\}$, $h(n) = \{1\}$

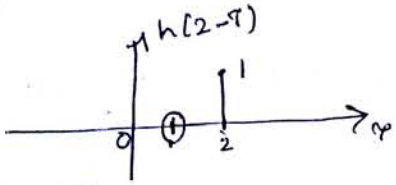
Sol $n_1 = 0$, $n_2 = 0$, so sequence starts at 0

$N_1 = 2$, $N_2 = 1$, $N_1 + N_2 - 1 = 2 + 1 - 1 = 2$





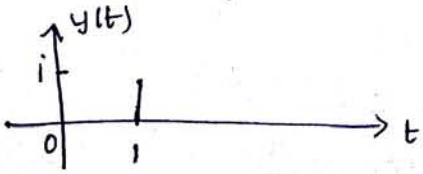




$y(0) = 1$

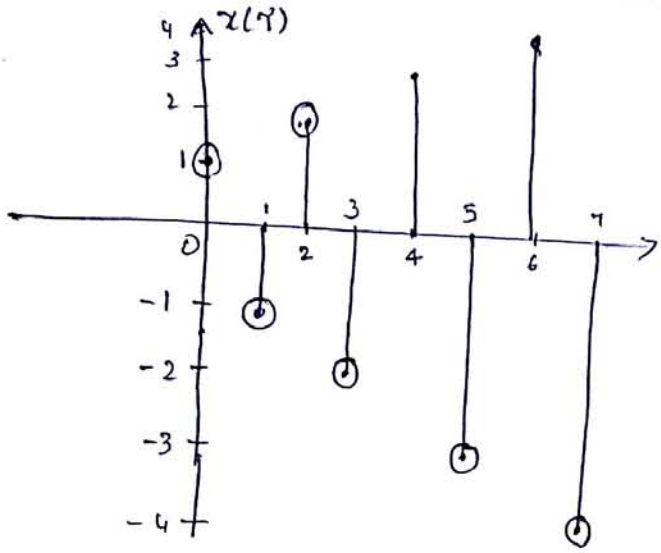
$y(1) = 0 + 0.5 \times 1 = 0.5$

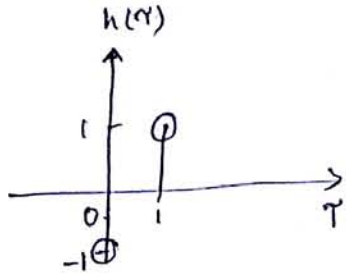
$y(2) = 0$.

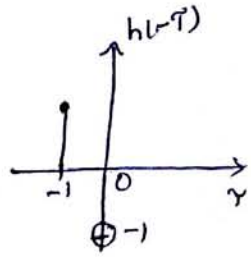


6) Perform the linear convolution of $x(n) = \{1, -1, 2, -2, 3, -3, 4, -4\}$ $h(n) = \{-1, 1\}$

Sol





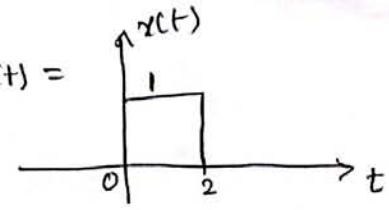


$n_1 = 0$, $n_2 = 0$ O/p sequence starts from 0

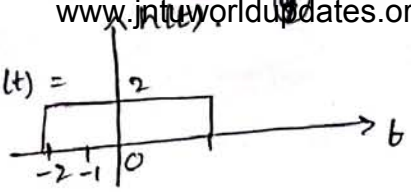
$N_1 = 8$, $N_2 = 2$, $N_1 + N_2 - 1 = 8 + 2 - 1 = 9$

$y(n) = \{-1, 2, -3, 4, -5, 6, -7, 8, 4\}$

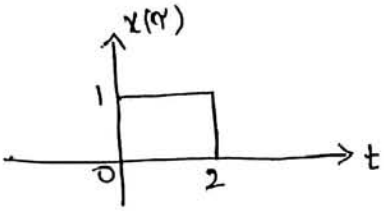
(7) Find the convolution of $x(t) =$

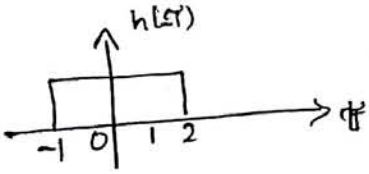


$h(t) =$

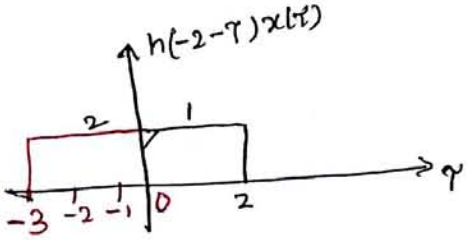


Sol



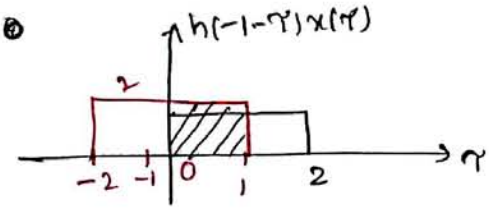


$y(-2) \Rightarrow 0$



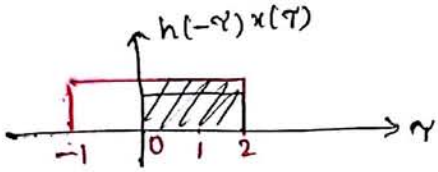
no overlap.

$y(-1) \Rightarrow$



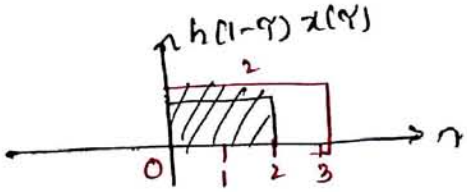
$\Rightarrow \int_0^1 2.1\, dt = 2.$

$y(0)$



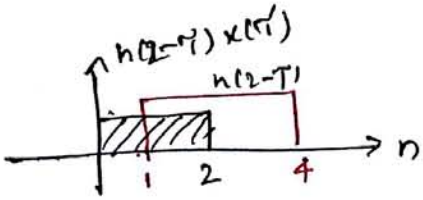
$\Rightarrow \int_0^2 2.1\, dt = 2[2-0] = 4$

$y(1) =$



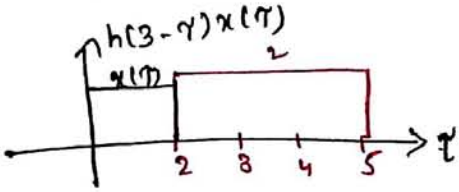
$\Rightarrow \int_0^2 2.1\, dt = 2[2-0] = 4$

$y(2) =$

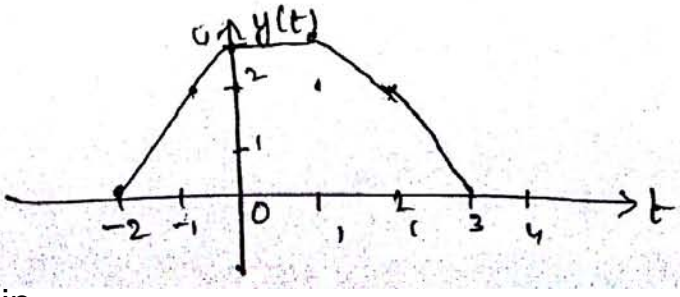


$\Rightarrow \int_1^2 2.1\, dt = 2[2-1] = 2$

$y(3) =$



$= \int$ no overlap $= 0$



PROPERTIES OF CONVOLUTION:

Let us consider two signals $x_1(t)$ and $x_2(t)$. The convolution of two signals is given by equation.

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

1. Commutative Property:

Convolution obeys commutative property.

$$x_1(t) * x_2(t) = x_2(t) * x_1(t).$$

Proof

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \rightarrow (1)$$

Let $t-\tau = p \Rightarrow \tau = t-p$ | when $\tau = -\infty$, $p = t+\infty = \infty$

$-d\tau = dp$ | when $\tau = \infty$, $p = t-\infty = -\infty$

substituting in eq (1) we get

$$x_1(t) * x_2(t) = -\int_{+\infty}^{-\infty} x_1(t-p)\, x_2(p)\, dp$$

$$= \int_{-\infty}^{\infty} x_2(p)\, x_1(t-p)\, dp$$

$$= x_2(t) * x_1(t)$$

2. Distributive Property:

$$x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t).$$

Proof

$$= x_1(t) * [x_2(t) + x_3(t)]$$

$$= x_1(t) * x_4(t) \quad \{\text{considering } x_4(t) = x_2(t) + x_3(t)\}$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, x_4(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, [x_2(t-\tau) + x_3(t-\tau)]\, d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau + \int_{-\infty}^{\infty} x_1(\tau)\, x_3(t-\tau)\, d\tau$$

$$= [x_1(t) * x_2(t)] + [x_1(t) * x_3(t)] \quad \text{RHS.}$$

(3) Associative property:

$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)]$ / $x_1(n) * [x_2(n) * x_3(n)] = [x_1(n) * x_2(n)] * x_3(n)$

Proof

Let $y_1(t) = x_1(t) * x_2(t)$

Let us replace by 'p'

$y_1(p) = x_1(p) * x_2(p)$

$= \sum_{m=-\infty}^{\infty} x_1(m) x_2(p-m)$

Let $y_2(t) = x_2(t) * x_3(t)$

$y_2(t) = \sum_{q=-\infty}^{\infty} x_1(q) x_2(t-q)$

$y_2(t-m) = \sum_{q=-\infty}^{\infty} x_1(q) x_2(t-m-q)$

where p, m, and q are dummy variables.

RHS:

$[x_1(t) * x_2(t)] * x_3(t)$

$= y_1(t) * x_3(t)$

$= \sum_{p=-\infty}^{\infty} y_1(p)$

Proof

Let $y_1(n) = x_1(n) * x_2(n)$

Let us replace n by p

$y_1(p) = x_1(p) * x_2(p)$

$= \sum_{m=-\infty}^{\infty} x_1(m) x_2(p-m)$

Let $y_2(n) = x_2(n) * x_3(n)$

$\therefore y_2(n) = \sum_{q=-\infty}^{\infty} x_1(q) x_2(n-q)$

$\therefore y_2(n-m) = \sum_{q=-\infty}^{\infty} x_1(q) x_2(n-q-m)$

LHS:

$[x_1(n) * x_2(n)] * x_3(n)$

$= y_1(n) * x_3(n)$

$= \sum_{p=-\infty}^{\infty} y_1(p) x_3(n-p)$

$= \sum_{p=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x_1(m) x_2(p-m) x_3(n-p)$

$= \sum_{m=-\infty}^{\infty} x_1(m) \sum_{p=-\infty}^{\infty} x_2(p-m) x_3(n-p).$

Let $p-m=q$

$\therefore p = m+q$

when $p=-\infty$, $q = p-m = -\infty-m = -\infty$

when $p=+\infty$ $q = p-m = +\infty-m = +\infty$.

On replacing (p-m) by 'q' and 'p' by 'q+m'

$= \sum_{m=-\infty}^{\infty} x_1(m) \sum_{q=-\infty}^{\infty} x_2(q) x_3(n-q-m)$

$= \sum_{m=-\infty}^{\infty} x_1(m) y_2(n-m)$

$= x_1(n) * y_2(n)$

$= x_1(n) * [x_2(n) * x_3(n)]$

$=$ RHS

(4) Shift property:

If $x_1(t) * x_2(t) = z(t)$ then $x_1(t) * x_2(t-T) = z(t-T)$

Proof

$$x_1(t) * x_2(t-T) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-T-\tau)\, d\tau$$

$$= z(t-T)$$

$\|^{ly}$ $x_1(t-T) * x_2(t) = z(t-T)$ and $x_1(t-T_1) * x_2(t-T_2) = z(t-T_1-T_2)$.

(5) Convolution with impulse:

Convolution of a signal $x(t)$ with unit impulse is the signal $x(t)$ itself.

$$x(t) * \delta(t) = x(t).$$

Proof

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

$$= x(t)$$

$\delta(t-\tau) = 1$ for $t = \tau$
$= 0$ otherwise

(6) Convolution with shifted impulse:

Convolution of a signal $x(t)$ with shifted impulse $\delta(t-t_0)$ is equal to $x(t-t_0)$

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

Proof

$$x(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau-t_0)\, d\tau$$

$$= x(\tau)\Big|_{\tau = t-t_0} = x(t-t_0).$$

(7) Convolution with unit step:

Convolution of a signal $x(t)$ with unit step signal $u(t)$ is given by

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Proof

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} x(\tau)\, d\tau$$

$\because u(t-\tau) = 1$ for $\tau < t \Rightarrow 0$ for $\tau > t$

Correlations:

→ Correlation is basically used to compare two signals or it is a measure of the degree to which two signals are similar.

→ Two types

(1) Cross-correlation

(2) Auto-correlation.

Cross Correlation:

The cross correlation between a pair of signals /sequences $x_1(t)$ and $y(t)$ is given by

$$\gamma_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\, y^*(t-\tau)\, dt \quad / \quad \gamma_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-l) \quad \text{where } l = 0, \pm 1, \pm 2, \cdots \rightarrow (1)$$

$$= \int_{-\infty}^{\infty} x(t+\tau)\, y^*(t)\, dt.$$

shift lag parameter.

→ The subscript xy indicates that $x(n)$ is the reference sequence that remains unshifted in time and $y(n)$ is shifted 'l' units in time w.r.to $x(n)$.

→ If we want to fix $y(n)$ and shift $x(n)$ then

$$\gamma_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)\, x(n-l)$$

$$= \sum_{n=-\infty}^{\infty} y(n+l)\, x(n) \rightarrow (2)$$

→ If time shift $l=0$ then we get

$$\gamma_{xy}(0) = \gamma_{yx}(0) = \sum_{n=-\infty}^{\infty} x(n)\, y(n).$$

Comparing eq(1) with eq(2) we find that

$$\gamma_{xy}(l) = \gamma_{yx}(-l)$$

↳ folded version of $\gamma_{xy}(l)$ abt $l=0$

We can rewrite the eq(1) as

$$\gamma_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\, y[-(l-n)]$$

$$= x(l) * y(-l)$$

Auto Correlation :

It is the correlation of a sequence within itself.

$$\gamma_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\, x(n-l)$$

(or)

$$\gamma_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n+l)\, x(n).$$

If time shift $l=0$, then

$$\gamma_{xx}(0) = \sum_{n=-\infty}^{\infty} x^2(n).$$

Auto Correlation Signal :

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

$$= \int_{-\infty}^{\infty} x(t+\tau)\, x^*(t)\, dt$$

For real values

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, dt$$

$$= \int_{-\infty}^{\infty} x(t+\tau)\, x(t)\, dt.$$

Properties of Cross Correlation function for energy signals.

1. For a real valued signals

$$R_{yx}(\tau) = R_{xy}(-\tau)$$

Proof

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t+\tau)\, y(t)\, dt$$

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} y(t+\tau)\, x(t)\, dt = \int_{-\infty}^{\infty} y(t)\, x(t-\tau)\, dt$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\, y(t)\, dt = R_{xy}(-\tau)$$

For complex valued signals $R_{yx}(\tau) = R_{xy}^*(-\tau)$.

(2) If $R_{xy}(0) = 0$ then $\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt = 0$. Then the sgls are said to be orthogonal over the entire time interval.

Cross correlation of periodic signals:

The cross correlation between two periodic signals $x(t)$ and $y(t)$ is given by

$$R_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, y^*(t-\tau)\, dt \quad \text{or} \quad \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, y^*(t-\tau)\, dt.$$

$$\text{|| ly} \quad R_{yx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} y(t)\, x^*(t-\tau)\, dt$$

Properties for periodic signals:

Property 1: The fourier transform of cross correlation is equal to multiplication of fourier transform of one signal and complex conjugate of F.T of other signal

$$R_{xy}(\tau) \longleftrightarrow \frac{1}{T_0^2} \sum_{k=-\infty}^{\infty} X_1(kf_0)\, X_2^*(kf_0)\, \delta(f-kf_0).$$

2: If cross correlation is executed at origin ($\tau = 0$)

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, y^*(t)\, dt = 0 \quad \text{i.e } R_{xy}(0) = 0. \text{ then sgls are}$$

said to be orthogonal

3: Cross correlation exhibits conjugate symmetry

$$R_{xy}(\tau) = R_{yx}^*(-\tau)$$

4. The cross correlation is not commutative

$$R_{xy}(\tau) \neq R_{yx}(\tau)$$

Properties for auto correlation function:

Property 1: The autocorrelation is an even fn of τ. That is $R_x(\tau) = R_x(-\tau)$

If $x(t)$ is real valued $\rightarrow$ It has even symmetry

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, dt \rightarrow R_{xx}(\tau) \int_{-\infty}^{\infty} x(t)\, x(t+\tau)\, dt$$

$$= \int_{-\infty}^{\infty} x(t+T)\, x(t)\, dt \Rightarrow$$

$t+T = x \Rightarrow dt = dx$.

$$= \int_{-\infty}^{\infty} f_1(x-T)\, f_1(x)\, dx$$

$$= \int_{-\infty}^{\infty} f_1(t-T)\, f_1(t)\, dt$$

$$= \phi_{11}(T)$$

Otherwise

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

$$R^*_{xx}(\tau) = \int_{-\infty}^{\infty} x^*(t)\, x(t+\tau)\, dt$$

$$\therefore R^*_{xx}(-\tau) = \int_{-\infty}^{\infty} x^*(t)\, x(t+\tau)\, dt$$

$$= R_{xx}(\tau).$$

For complex $\rightarrow$ It has hermitian symmetry. changing the variable x to t, we

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-T)\, dt$$

$$= \int_{-\infty}^{\infty} x(t+T)\, x^*(t)\, dt.$$

i.e $R_{xx}(-\tau) = R^*_{xx}(\tau)$.

Property 2:

The autocorrelation fn is bounded by its value at the origin. That is

$$R_x(0) \geq |R_x(\tau)| \text{ for any } \tau$$

$\rightarrow$ The largest value occurs at $\tau = 0$ of autocorrelation fn.

proof Consider a finite energy signal $x(t)$.

Consider the fn $x(t)$ & $x(t+\tau)$

$$[x(t) + x(t+\tau)]^2 \geq 0$$

$$x^2(t) + x^2(t+\tau) \pm 2x(t)x(t+\tau) \geq 0$$

$$x^2(t) + x^2(t+\tau) \geq \pm 2x(t)\, x(t+\tau)$$

Integrating on b.s.

$$\therefore \int_{-\infty}^{\infty} |x(t)|^2 dt + \int_{-\infty}^{\infty} |x(t+\tau)|^2 dt \geq 2\int_{-\infty}^{\infty} x(t)\, x(t+\tau)\, dt$$

$$\therefore E + E \geq 2R_{xx}(\tau)$$

$$E \geq R_{xx}(\tau)$$

$$\text{or } R(0) \geq R_{xx}(\tau)$$

Other method:

$$y(t) = \int_{-\infty}^{\infty} |x(t+\tau) - \underbrace{a}_{\text{finite value}} x(t)|^2 dt. \text{ Obviously } |y(t)| > 0$$

$$y(t) = \int_{-\infty}^{\infty} |x(t+\tau) - a\,x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} |x(t+\tau)|^2 dt + |a|^2 \int_{-\infty}^{\infty} |x(t)|^2 dt - 2|a| \left| \int_{-\infty}^{\infty} x(t+\tau)\, x(t) \right| dt$$

$$R_{xx}(0) = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t+\tau)|^2 dt$$

$$y(t) \geq [1 + |a|^2]\, R_{xx}(0) - 2|a|\, |R_{xx}(\tau)|$$

For $|a| = 1$

$$2R_{xx}(0) \geq 2|R_x(\tau)| \quad \text{since } |y(t)| > 0.$$

$$\Rightarrow R_x(0) > R_x(\tau)$$

Property 3:

The value of autocorrelation fn at $\tau=0$ is equal to energy of the signal

$$E = R_{xx}(0) = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Proof

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

$$R_{xx}(0) = \int_{-\infty}^{\infty} x(t) \cdot x^*(t)\, dt \Rightarrow \int_{-\infty}^{\infty} |x(t)|^2 dt = E$$

Autocorrelation of power signals:

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^*(t-T)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t+T)\, x^*(t)\, dt$$

by putting $\tau=0$

$$R_{xx}(0) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$\boxed{R_{xx}(0) = P}$$

$\therefore$ The autocorrelation fn at origin is equal to average power of the signal.

For periodic signals:

(1) If $R_{xx}(\tau)$ and $R^*_{xx}(-\tau)$ are complex conjugate of each other then $R(\tau) = R^*(-\tau)$ called as conjugate property.

Proof

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^*(t-\tau)\, dt$$

$$R^*_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^*(t)\, x(t-\tau)\, dt$$

$$R^*_{xx}(-\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^*(t)\, x(t+\tau)\, dt = R(\tau)$$

Correlation :

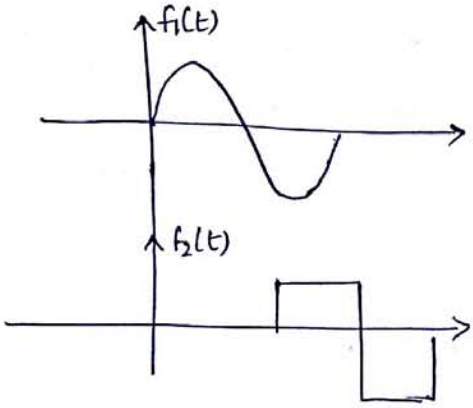
→ correlation means comparison

There are two types of correlations

① cross correlation

(2) Auto correlation.

Cross correlation : It is the measure of similarity between one waveform and time delayed version of the another waveform.

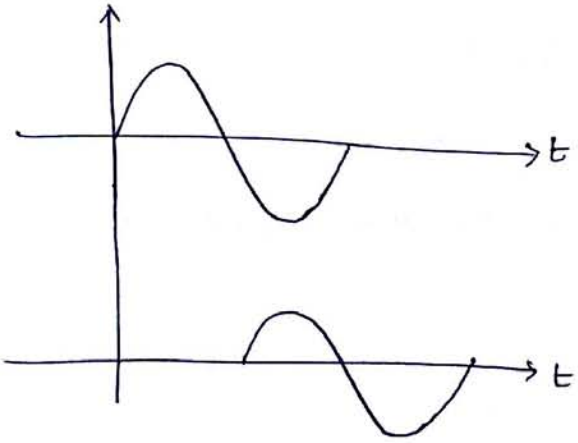


Equation of cross correlation is

$$R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t)\, f_2^*(t-\tau)\, dt$$

Auto correlation : It is the measure of similarity between one waveform and time delayed version of the same waveform.

Ex:



Equation of auto correlation is

$$R_{11}(\tau) = \int_{-\infty}^{\infty} f_1(t)\, f_1^*(t-\tau)\, dt.$$

APPLICATION OF CORRELATION:

→ It is used in the determination of signal that are contaminated with noise.

CROSS CORRELATION FOR ENERGY SIGNALS:

$$E = \int_{-\infty}^{\infty} f^2(t)\, dt \rightarrow \text{cross} \rightarrow \text{two fns}$$

Let $f_1(t)$ & $f_2(t)$ be two energy signals then cross co-relation b/w them is defined as

$$R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t)\, f_2^*(t-\tau)\, dt$$

$$R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t+\tau)\, f_2^*(t)\, dt$$

Here τ is delay parameter or scanning parameter or scaling parameter

→ From the above two eqns cross correlation function obtained by shifting $f_2(t)$ in +ve direction by an amount τ is equal to the cross correlation function obtained by shifting $f_1(t)$ in −ve direction by an amount τ.

$$R_{21}(\tau) = \int_{-\infty}^{\infty} f_2(t)\, f_1^*(t-\tau)\, dt$$

$$R_{21}(\tau) = +\int_{-\infty}^{\infty} f_2(t+\tau)\, f_1^*(t)\, dt$$

By comparing with the convolution integral we can define cross correlation function of $f_1(t)$ & $f_2(t)$ as

$$R_{12}(\tau) = f_1(t) \otimes f_2(-t)$$

→ If $f_1(t)$ & $f_2(t)$ are even fns then cross correlation function becomes equal to convolution.

Differences between convolution & correlation

convolution	correlation
→ $f_1(t) \otimes f_2(t) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau$ τ = dummy variable	→ $R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t)\, f_2^*(t-\tau)\, dt$ t = dummy variable.
→ convolution is a fn of physical time 't'	→ function of τ
→ It obeys commutation law $f_1(t) \otimes f_2(t) = f_2(t) \otimes f_1(t)$	→ It does not obey commutation Law. $R_{12}(\tau) \neq R_{21}(\tau)$
→ It is used to evaluate the response of the system for arbitary i/p $f_1(t) \otimes f_2(t) = F_1(\omega)\, F_2(\omega)$	→ It is used to eliminate noise from the signals.

PROPERTIES OF CROSS CORRELATION:

Property 1: Proof: The F.T of cross correlation function of two signals is equal to product of FT of one signal and complex conjugate F.T of other signal.

$$F[R_{12}(\tau)] = X_1(f)\, X_2^*(f)$$

Proof

$$X_1(f) = \int_{-\infty}^{\infty} x_1(t)\, e^{-j2\pi ft}\, dt \rightarrow (1)$$

$$X_2(f) = \int_{-\infty}^{\infty} x_2(t)\, e^{-j2\pi ft}\, dt \rightarrow (2)$$

put $t = t - \tau$

$dt = -d\tau$

$t = t - \tau$

$t = t + \infty = \infty$

$t = t - \infty = -\infty$

$\frac{dt}{d\tau} = 0 - \frac{d\tau}{d\tau}$

$\frac{dt}{d\tau} = -1$

$\Rightarrow dt = -d\tau$

$$X_2(f) = \int_{+\infty}^{-\infty} x_2(t-\tau)\, e^{-j2\pi ft} \cdot e^{j2\pi f\tau}\, (-d\tau)$$

$$X_2(f) = \int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j2\pi ft} \cdot e^{j2\pi f\tau}\, d\tau$$

Applying conjugate on both sides

$$X_2^*(f) = \int_{-\infty}^{\infty} x_2^*(t-\tau)\, e^{j2\pi ft} \cdot e^{-j2\pi f\tau}\, d\tau \rightarrow (3)$$

Now multiplying the eq(1) & (3) we get

$$X_1(f)\, X_2^*(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, e^{-j2\pi f\tau}\, dt\, d\tau$$

$$= \int_{-\infty}^{\infty} R_{12}(\tau)\, e^{-j2\pi f\tau}\, d\tau \qquad \left[\because R_{12} = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, dt\right]$$

$$\boxed{X_1(f)\, X_2^*(f) = F[R_{12}(\tau)]}$$

(2) The cross correlation fn is zero at origin i.e at $\tau = 0$,

$$R_{12}(\tau) = R_{12}(0) = 0$$

proof: Consider $R_{12}(\tau) = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_1(t)\, x_2^*(t-\tau)\, dt$

$$R_{12}(0) = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_1(t)\, x_2^*(t)\, dt$$

$$= 0 \qquad (\because x_1(t) \,\&\, x_2(t) \text{ are periodic})$$

3) The cross correlation fn satisfies conjugate symmetry property i.e $R_{12}(\tau) = R_{21}^*(-\tau)$

Proof

$$R_{12}(\tau) = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_1(t)\, x_2^*(t-\tau)\, dt \rightarrow (1)$$

$$R_{21}(\tau) = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_2(t)\, x_1^*(t-\tau)\, dt \rightarrow (2)$$

Putting $\tau = -\tau$

$$R_{21}(-\tau) = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_2(t)\, x_1^*(t+\tau)\, dt$$

Applying conjugate on both sides, we get

$$R_{21}^*(-\tau) = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_2^*(t)\, x_1(t+\tau)\, dt$$

{Now put substitute $t = t-\tau$
$dt = dt$ }

$$= \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_2^*(t-\tau)\, x_1(t)\, dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_1(t)\, x_2^*(t-\tau)\, dt$$

$$= R_{12}(\tau)$$

$$\boxed{\therefore\ R_{21}^*(-\tau) = R_{12}(\tau)}$$

(4) The cross correlation fn does not satisfy commutative property i.e $R_{12}(\tau) \neq R_{21}(\tau)$

Proof

$$R_{12}(\tau) = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_1(t)\, x_2^*(t-\tau)\, dt$$

$$R_{21}(\tau) = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} x_2(t)\, x_1^*(t-\tau)\, dt$$

As above eqn are not same

$$R_{12}(\tau) \neq R_{21}(\tau)$$

(5) The cross correlation for energy signals is given by

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, dt$$

(or)

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t)\, x_1^*(t-\tau)\, dt$$

is their cross

(8) Consider two energy signals $g_1(t)$ & $g_2(t)$, which may be complex valued. Show that $\int_{-\infty}^{\infty} |g_1(t) - g_2(t)|^2 dt = R_{11}(0) + R_{22}(0) - 2\,Re[R_{12}(0)]$ where $R_{11}(\lambda)$ and $R_{22}(\lambda)$ are autocorrelation fn of $g_1(t)$ & $g_2(t)$ & $R_{12}(\lambda)$

PROBLEMS ON AUTOCORRELATION AND CROSS CORRELATION

(1) Determine the cross correlation fn $R_{xy}(\lambda)$ $\rightarrow R_{12}(\lambda)$ of two signal $g_1(t)$ and $g_2(t)$ defined by

$$g_1(t) = \begin{cases} A\cos(2\pi f_1 t + \theta_1) & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases} \qquad g_2(t) = \begin{cases} A\cos(2\pi f_2 t + \theta_2) & 0 \leq t < T \\ 0 & \text{elsewhere} \end{cases}$$

How does varying the frequency difference $|f_1 - f_2|$ affect this cross correlation fn.

Sol Cross correlation is given as

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, dt$$

or

$$R_{12}(\tau) = \int_{-\infty}^{\infty} g_1(t)\, g_2^*(t-\lambda)\, dt$$

$$= \int_0^T A\cos(2\pi f_1 t + \theta_1) \cdot A\cos(2\pi f_2 (t-\lambda) + \theta_2)\, dt$$

$$= \int_0^T A\cos(2\pi f_1 t + \theta_1) \cdot A\cos(2\pi f_2 t - 2\pi f_2 \lambda + \theta_2)\, dt$$

$$\left\{ \because \cos(A-B) + \cos(A+B) \right]$$

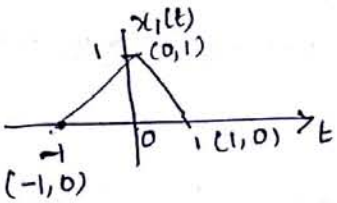
$$= \frac{A^2}{2} \int_0^T \left\{ \cos(2\pi f_1 t + \theta_1 - 2\pi f_2 t - \theta_2 + 2\pi f_2 \lambda) + \cos(2\pi f_1 t + \theta_1 + 2\pi f_2 t - 2\pi f_2 \lambda + \theta_2) \right\} dt$$

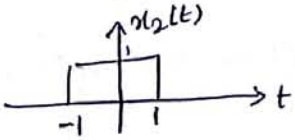
$$= \frac{A^2}{2} \left\{ \frac{\sin(2\pi f_1 t + \theta_1 - 2\pi f_2 t - \theta_2 + 2\pi f_2 \lambda)}{2\pi(f_1 - f_2)} \right]_0^T + \frac{\sin[2\pi(f_1+f_2)t + \theta_1 + \theta_2 - 2\pi f_2 \lambda]}{2\pi(f_1+f_2)} \Bigg]_0^T$$

$$= \frac{A^2}{2} \left[\frac{\sin(2\pi(f_1-f_2)T + \theta_1 - \theta_2 + 2\pi f_2 \lambda) - \sin[\theta_1 - \theta_2 + 2\pi f_2 \lambda]}{2\pi(f_1-f_2)} \right.$$

$$\left. + \frac{\sin[(2\pi(f_1+f_2)T + \theta_1 + \theta_2 - 2\pi f_2 \lambda) - \sin[\theta_1 + \theta_2 - 2\pi f_2 \lambda]}{2\pi(f_1+f_2)} \right]$$

2) Find the cross correlation b/w triangular and gate pulse as shown in fig.





Sol

$x_1(t)$ $\quad (x_1, y_1) = (-1, 0), (x_2, y_2) = (0, 1)$

$y = x_1(t)$, $x = t$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$x_1(t) = t + 1$$

$$x_1(t) = \begin{cases} 1 + t & \text{for } -1 \le t \le 0 \\ 1 - t & \text{for } 0 \le t \le 1 \end{cases}$$

$x_2(t) = 1$ for $-1 \le t \le 1$

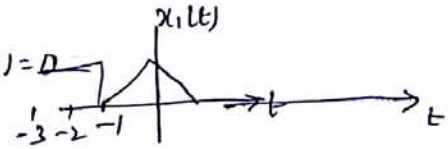
$x_2(t - \tau) = 1$ for $-1 \le t - \tau \le 1$

i.e $\tau - 1 \le t \le \tau + 1$

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2(t - \tau)\, dt.$$

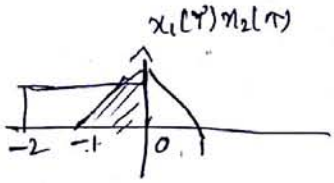
Case i; $\tau < -2$

There is no overlap b/w $x_1(t)$ and $x_2(t)$ $\therefore$ $R_{12}(\tau) = 0$



Case (ii) $-2 \le \tau \le -1$

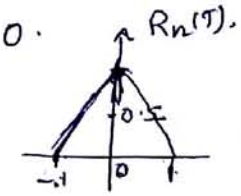
$$R_{12}(\tau) = \int_{-1}^{0} (t + 1)\, dt$$



$$= \cancel{\frac{t^2}{2} + t \Big|_{-1}^{0}}$$

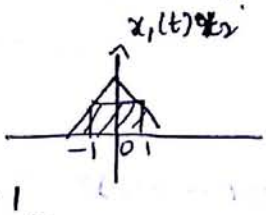
$\frac{bh}{2} \Rightarrow \frac{1}{2} \times 1 \times 1 = 0.5$ between -1 to 0.

$$= \frac{1}{2} + 1 =$$



case (iii)

$$R_{12}(\tau) = \frac{1}{2} \times 1 \times 1 +$$



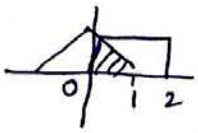
between -1 to 1

$$\frac{1}{2} \times 1 \times 1 = 1$$

Case (iv)

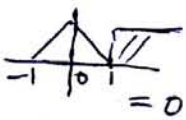
$$= \frac{1}{2} \times 1 \times 1$$

$$= 0.5$$



between 0 to 1

Case (v)



$= 0.$

Energy Density Spectrum:

Spectral density: It is the distribution of power or energy of a signal per unit bandwidth as a function of frequency.

Energy and power Signals:

→ Signals with finite energy i.e $0<E<\infty$ and $P=0$ are called energy signals.

e.g: aperiodic signals like pulse

→ Signals with finite average power i.e $0<P<\infty$ and $E=\infty$ are called power sgls i.e periodic signals.

The energy of a signal $x(t)$ is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt.$$

Parseval's Theorem for energy signal: / Rayleigh energy theorem:

It defines the energy of a signal in terms of fourier transform.

$$\text{i.e } E = \int_{-\infty}^{\infty} |X(f)|^2 df.$$

Proof $x(t) \longleftrightarrow X(f)$. Let $x^*(t)$ be conjugate of $x(t)$ such that

$$x^*(t) \longleftrightarrow X^*(-f)$$

Energy of the signal $x(t)$ is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt = \int_{-\infty}^{\infty} x^*(t)\, x(t)\, dt.$$

Replacing $x(t)$ in terms of inverse fourier transform $X(f)$

$$E = \int_{-\infty}^{\infty} x^*(t) \left\{ \int_{-\infty}^{\infty} X(f)\, e^{j\omega t}\, df \right\} dt$$

Interchanging the order of integration.

$$E = \int_{-\infty}^{\infty} X(f) \left\{ \int_{-\infty}^{\infty} x^*(t)\, e^{j\omega t}\, dt \right\} df$$

$$E = \int_{-\infty}^{\infty} X(f)\, X^*(f)\, df \Rightarrow \int_{-\infty}^{\infty} |X(f)|^2 df.$$

Energy Spectral density: It is the distribution of energy of signal in frequency domain. which is also called as energy density spectrum (ESD or ED) given by

$$\psi(f) = |X(f)|^2 \rightarrow (1)$$

Let $x(t)$ and $y(t)$ be the input and output of a linear system. i.e $x(t) \longleftrightarrow X(f)$ and $y(t) \longleftrightarrow Y(f)$ and $H(f)$ be system transfer function.

$$Y(f) = H(f)X(f) \rightarrow (2)$$

Using Eq (1) we can write as

$$\psi_x(f) = |X(f)|^2$$

$$\psi_y(f) = |Y(f)|^2.$$

$$\psi_y(f) = |Y(f)|^2 = |H(f)|^2 |X(f)|^2 = |H(f)|^2 \psi_x(f).$$

$$\boxed{\psi_y(f) = |H(f)|^2 \psi_x(f)}$$

ESD of the output is the product of ESD of input and square of the magnitude of transfer function.

Power Density Spectrum:

Average power is defined as

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

But power P is defined as

$$P = \overline{x^2(t)} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt.$$

Parseval's Power Theorem: It defines the power of a signals in terms of its fourier series coefficients.

$$P = \sum_{n=-\infty}^{\infty} |F_n|^2$$

Proof : Consider a function $x(t)$

$$|x(t)|^2 = x(t)\, x^*(t) \rightarrow (1)$$

$x^*(t) \rightarrow$ conjugate of $x(t)$.

Average power of $x(t)$ for one cycle is

$$P = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, x^*(t)\, dt \rightarrow (2)$$

we have exponential Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \rightarrow (3)$$

Replacing $x(t)$ of eq(2) by eq(3) we get

$$P = \frac{1}{T}\int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}\, x^*(t)\, dt$$

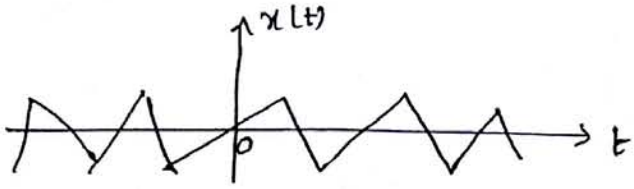
$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} F_n \int_{-T/2}^{T/2} x^*(t)\, e^{jn\omega_0 t}\, dt = \sum_{n=-\infty}^{\infty} F_n \cdot \frac{1}{T}\int_{-T/2}^{T/2} x^*(t)\, e^{jn\omega_0 t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} F_n \cdot F_n^* = \sum_{n=-\infty}^{\infty} |F_n|^2 = P.$$

↳ parsevals power theorem.

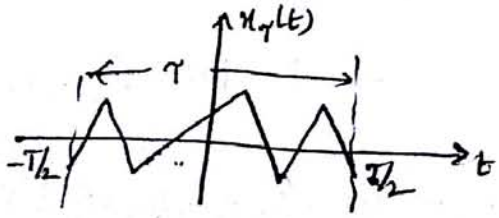
Power Spectral Density (PSD):

The distribution of average power of the signal in frequency domain is called power spectral density or power density spectrum (PSD or PD)



Let us assume that signal is zero outside the interval $(\tau/2)$.

$$x_\tau(t) = \begin{cases} x(t) & |t| < \tau/2 \\ 0 & \text{elsewhere} \end{cases}$$



The signal $x_T(t)$ is of finite duration T and hence it energy signal with energy E given by

$$E = \int_{-\infty}^{\infty} |x_T(t)|^2 dt = \int_{-\infty}^{\infty} |X_T(f)|^2 df$$

where $x_T(t) \longleftrightarrow X_T(f)$.

As $x(t)$ over interval $(-T/2, T/2)$ is same as $x_T(t)$ over the interval $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} |x_T(t)|^2 dt = \int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$\therefore \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 = \frac{1}{T}\int_{-\infty}^{\infty} |X_T(f)|^2 df \rightarrow (1)$$

If $T \rightarrow \infty$, the left hand side of eq (1) represents average power P

$$P = \int_{-\infty}^{\infty} \lim_{T\rightarrow\infty} \frac{|X_T(f)|^2}{T} df$$

If $T \rightarrow \infty$, $\frac{|X_T(f)|^2}{T}$ approaches finite value denoted by $S(f)$ or $S(\omega)$

$$S(f) = \lim_{T\rightarrow\infty} \frac{|X_T(f)|^2}{T}$$

Average power

$$P = \overline{x^2(t)} = \int_{-\infty}^{\infty} S(f)\, df = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, d\omega.$$

→ The PSD of periodic function is given by

$$S(f) = \sum_{n=-\infty}^{\infty} |F_n|^2 \delta(f - n f_0)$$

$$\text{ll}^{ly} \quad S(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |F_n|^2 \delta(\omega - n\omega_0).$$

I/p and o/p relation of linear system in terms of PSD is given by

$$S_y(f) = |H(f)|^2 \cdot S_x(f).$$

Comparsion of ESD & PSD

ESD	PSD
(1) It gives the distribution of energy of a signal in frequency	(1) It gives the distribution of power of signal in frequency domain
(2) It is given by $\psi(f) = \lvert X(f)\rvert^2$	(2) It is given by $S(f) = \lim_{T\to\infty} \frac{\lvert X_T(f)\rvert^2}{T}$
(3) Total energy is given by $E = \int_{-\infty}^{\infty} \psi(f)\,df$	(3) Total power is given by $P = \int_{-\infty}^{\infty} S(f)\,df$
(4) The autocorrelation for an energy signal and its ESD form a Fourier transform pair $R(\tau) \longleftrightarrow \psi(f)$	(4) The autocorrelation for a power signal and its PSD form a Fourier transform pair. $R(\tau) \longleftrightarrow S(f)$

RELATION BETWEEN AUTOCORRELATION AND SPECTRAL DENSITIES

1. The autocorrelation function $R(\tau)$ of an energy signal and its energy spectral density (ESD), $\psi(f)$ forms a fourier transform pair,

$$R(\tau) \longleftrightarrow \psi(f)$$

proof: Cross correlation of two energy signals $x(t)$ and $y(t)$ is given as

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} X(f)\, Y^*(f)\, e^{j\omega\tau}\, df.$$

If both functions are same, then auto correlation is given by

$$R(\tau) = \int_{-\infty}^{\infty} X(f)\, X^*(f)\, e^{j\omega\tau}\, df = \int_{-\infty}^{\infty} \lvert X(f)\rvert^2 e^{j\omega\tau}\, df$$

$$= F^{-1}\left[\lvert F(f)\rvert^2\right]$$

But $\lvert F(f)\rvert^2 = \psi(f)$

$$\therefore R(\tau) = F^{-1}[\psi(f)] \Rightarrow F[R(\tau)] = \psi(f)$$

$$R(\tau) \longleftrightarrow \psi(f)$$

2) The autocorrelation function $R(\tau)$ and power spectral density (PSD), $S(f)$ of a power signal form a fourier transform pair.

$$R(\tau) \longleftrightarrow S(f)$$

proof

Autocorrelation function of power $x(t)$ in terms of fourier series coefficients is given as

$$R(\tau) = \sum_{n=-\infty}^{\infty} \underbrace{X_n X_{-n}}_{\text{exponential fourier series coefficients}} e^{jn\omega_0 \tau}$$

$$R(\tau) = \sum_{n=-\infty}^{\infty} |X_n|^2 e^{jn\omega_0 \tau}$$

Taking Fourier transform.

$$F[R(\tau)] = \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} |X_n|^2 e^{jn\omega_0 \tau} \right] e^{-j\omega\tau} d\tau$$

Interchanging the order of integration & summation, we get

$$F[R(\tau)] = \sum_{n=-\infty}^{\infty} |X_n|^2 \int_{-\infty}^{\infty} e^{-j\tau(\omega - n\omega_0)} d\tau$$

$$= 2\pi \sum_{n=-\infty}^{\infty} |X_n|^2 \cdot \delta(\omega - n\omega_0) = \sum_{n=-\infty}^{\infty} |X_n|^2 \delta(f - nf_0).$$

The RHS is the PSD $S(\omega)$ or $S(f)$ of periodic function $x(t)$

$$\therefore F[R(\tau)] = S(f)$$

$$R(\tau) = F^{-1}[S(f)]$$

$$\therefore R(\tau) \longleftrightarrow S(f).$$

RELATION BETWEEN CONVOLUTION AND CORRELATION:

(1) In correlation, physical time 't' plays the role of dummy variable & it appears after solving the integral but in convolution delay parameter τ plays the role of dummy variable.

2) Correlation $R_{xy}(\tau)$ is a function of delay parameter τ, whereas convolution is a function of time.

3) Correlation can be obtained by convolving $x(t)$ & $y^*(t)$

4) Convolution does not depend on which function is being shifted whereas correlation does i.e convolution is commutative.

(7) The convolution and correlation are identical for even signals.

Proof For two signals $x_1(t)$ and $x_2(t)$

The definition for correlation is given by

$$R_{12} = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt \quad \rightarrow (1)$$

$$R_{21} = \int_{-\infty}^{\infty} x_2(t)\, x_1(t-\tau)\, dt \quad \rightarrow (2)$$

The definition for convolution

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \quad \rightarrow (3)$$

$$= \int_{-\infty}^{\infty} x_2(\tau)\, x_1(t-\tau)\, d\tau \quad \rightarrow (4)$$

Taking eq (1) & eq (3)

$$R_{12} = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt$$

Replacing dummy variable 't' by 'p' we get

$$= \int_{-\infty}^{\infty} x_1(p)\, x_2(p-\tau)\, dp$$

Since it is a even signal i.e $x(t) = x(-t)$

$$= \int_{-\infty}^{\infty} x_1(p)\, x_2(-(\tau-p))\, dp = S$$

$$S = \int_{-\infty}^{\infty} x_1(p)\, x_2(\tau-p)\, dp$$

Replace dummy variable τ by t

$$S = \int_{-\infty}^{\infty} x_1(p)\, x_2(t-p)\, dp$$

Again replacing the variable p by τ

$$S = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$= x_1(t) * x_2(t).$$

Extension of Autocorrelation Property.

(4) Fourier transform of auto correlation yields π times the energy density spectrum of $f_1(t)$ $\quad |F_1(\omega)|^2 = \int_{-\infty}^{\infty} R_{11}(T)\, e^{-j\omega t} dT = \pi S_1(\omega)$

proof

From definition of FT, FT of $R_{11}(T)$ is

$$\int_{-\infty}^{\infty} R_{11}(T) e^{-j\omega T} dT = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(t)\cdot f_1(t-T)\, e^{-j\omega T}\, dt\, dT$$

$$= \int_{-\infty}^{\infty} f_1(t)\, e^{-j\omega t}\, dt \int_{-\infty}^{\infty} f_1(t-T)\, e^{j\omega(t-T)}\, dT$$

putting $t-T = -x \Rightarrow dT = dx$

$$= F_1(\omega)\int_{-\infty}^{\infty} f_1(-x)\, e^{-j\omega x}\, dx$$

$$= F_1(\omega)\, F_1(-\omega)$$

$$= |F_1(\omega)|^2 = \pi S_1(\omega).$$

(5) Fourier transform of R_{12} (cross correlation fn) is $F_1(\omega)F_2(-\omega) \longleftrightarrow F_1(\omega)F_2(-\omega)$

proof Fourier transform of $f_1(t)$ & $f_2(t)$ are

$f_1(t) \longleftrightarrow F_1(\omega)$, $f_2(t) \longleftrightarrow F_2(\omega)$ $\|^{ly}$ $f_1(-t) \longleftrightarrow F_1(-\omega)$ & $f_2(-t) \longleftrightarrow F_2(-\omega)$

$$R_{12}(T) = f_1(t) * f_2(-t).$$

$\therefore$ Fourier transform of $f_1(t) * f_2(-t)$ is $F_1(\omega)F_2(-\omega)$.

$$\therefore\ R_{12}(T) = f_1(t) * f_2(-t) \longleftrightarrow F_1(\omega)\cdot F_2(-\omega).$$

(6) Graphically $R_{12}(T)$ is same as $R_{21}(T)$ where it is folded back about the vertical axis at $T=0$. $R_{12}(T) = R_{21}(-T)$.

proof

$$R_{12}(T) = \int_{-\infty}^{\infty} f_1(t)\cdot f_2(t-T)\, dt = \int_{-\infty}^{\infty} f_1(t+T)\, f_2(t)\, dt$$

$$R_{21}(T) = \int_{-\infty}^{\infty} f_2(t)\cdot f_1(t-T)\, dt$$

$$R_{21}(-T) = \int_{-\infty}^{\infty} f_2(t)\cdot f_1(t-(-T))\, dt = \int_{-\infty}^{\infty} f_2(t)\, f_1(t+T)\, dt = R_{12}(T)$$

ESD (Energy Spectral Density):

→ Gives the distribution of energy of the signal in the frequency domain.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

we know

$$\omega = 2\pi f$$

$$d\omega = 2\pi df$$

$$\therefore E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(f)|^2 \cdot 2\pi df = \int_{-\infty}^{\infty} |X(f)|^2 df \quad \text{coz } |X(2\pi f)| \text{ is written as } X(f). \quad \hookrightarrow (1)$$

where $|X(f)| \rightarrow$ amplitude Spectrum.

If we denote $|X(f)|^2$ by $\psi(f)$

$$\therefore \text{ESD}: \psi(f) = |X(f)|^2 \rightarrow (2)$$

putting (2) in eq (1)

$$E = \int_{-\infty}^{\infty} \psi(f) df.$$

→ total area under the curve $\psi(f)$.

↓ total energy of the sgl.

→ $\psi(f)$ represents → Energy spectral density of sgl $x(t)$ in joules per hertz.

Effect of Systems on ESD

Let the ESD of $x(t)$ be $\psi_x(f)$ and $y(t)$ be $\psi_y(f)$. The signal $x(t)$ is applied at the input of LTI system and $y(t)$ is obtained at the output.

→ Let LTI system to be an ideal filter which has pass band from f_L to f_H. i.e only signal will be passed without any effect from f_L to f_H.

we know that $E = \int_{-\infty}^{\infty} \psi(f)\, df$

Energy at the output will be

$$E_y = \int_{-\infty}^{\infty} \psi_y(f)\, df.$$

→ If $\psi_y(f)$ is symmetric for positive & negative values of 'f', then

$$E_y = 2\int_{0}^{\infty} \psi_y(f)\, df.$$

$$E_y = 2\int_{f_L}^{f_H} \psi_y(f)\, df$$

$$= 2\int_{f_L}^{f_H} |Y(f)|^2 df.$$

we know $Y(\omega) = H(\omega)X(\omega)$.

$$E_y = 2\int_{f_L}^{f_H} |H(f)X(f)|^2 df$$

$$= 2\int_{f_L}^{f_H} |H(f)|^2 |X(f)|^2 df$$

$$= 2\int_{f_L}^{f_H} |H(f)|^2 \psi_x(f)\, df$$

→ The filter passes all the frequencies b/w f_L & f_H i.e $H(f) = 1$ for $f_L \leq f \leq f_H$.

$$E_y = 2\int_{f_L}^{f_H} \psi_x(f)\, df$$

energy of the o/p sgl. in terms of ESD of i/p sgl.

RELATION BETWEEN CORRELATION AND CONVOLUTION:

The convolution of $f_1(t)$ and $f_2(-t)$ by $\rho_{12}(t)$, we have

$$\rho_{12}(t) = f_1(t) * f_2(-t)$$
$$= \int_{-\infty}^{\infty} f_1(\tau) f_2(\tau - t)\, d\tau$$

The dummy variable τ in the above integral may be replaced by another variable x.

$$\rho_{12}(t) = \int_{-\infty}^{\infty} f_1(x) \cdot f_2(x-t)\, dx.$$

changing the variable from t to τ, we get

$$\rho_{12}(t) = \int_{-\infty}^{\infty} f_1(x)\, f_2(x-\tau)\, dx$$
$$= \phi_{12}(\tau)$$

Hence $\phi_{12}(\tau) = f_1(t) * f_2(-t)\big|_{t=\tau} = \rho_{12}(\tau)$

$\|^{ly}$ $\phi_{21}(\tau) = f_1(-t) * f_2(t)\big|_{t=\tau} = \rho_{21}(\tau).$

and $\phi_{11}(\tau) = f_1(t) * f_1(-t)\big|_{t=\tau} = \rho_{11}(\tau)$

DETECTION OF PERIODIC SIGNALS IN THE PRESENCE OF NOISE BY CORRELATION:

→ Now we consider that periodic signals are affected by noise that finds the applications in the detection of radar and sonar signals, periodic component in brain waves and cyclical component in ocean wave analysis.

→ If $s(t)$ is periodic signal and $n(t)$ represents the noise signal, then

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} s(t)\, n(t-\tau)\, dt = 0 \text{ for all } \tau.$$

$\phi_{sn}(\tau) = 0$

↙

crosscorrelation fn

Detection by Autocorrelation:

Let $s(t)$ be a periodic signal mixed with noise signal $n(t)$. Then the received sgl $f(t)$ is $[s(t)+n(t)]$.

→ Let $\bar{\varphi}_{ff}(\tau)$, $\bar{\varphi}_{ss}(\tau)$, $\bar{\varphi}_{nn}(\tau)$ denote the autocorrelation functions of $f(t), s(t)$ and $n(t)$ respectively.

$$\bar{\varphi}_{ff}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, f(t-\tau)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [s(t)+n(t)][s(t-\tau)+n(t-\tau)]\, dt$$

$$= \bar{\varphi}_{ss}(\tau) + \bar{\varphi}_{nn}(\tau) + \bar{\varphi}_{sn}(\tau) + \bar{\varphi}_{ns}(\tau)$$

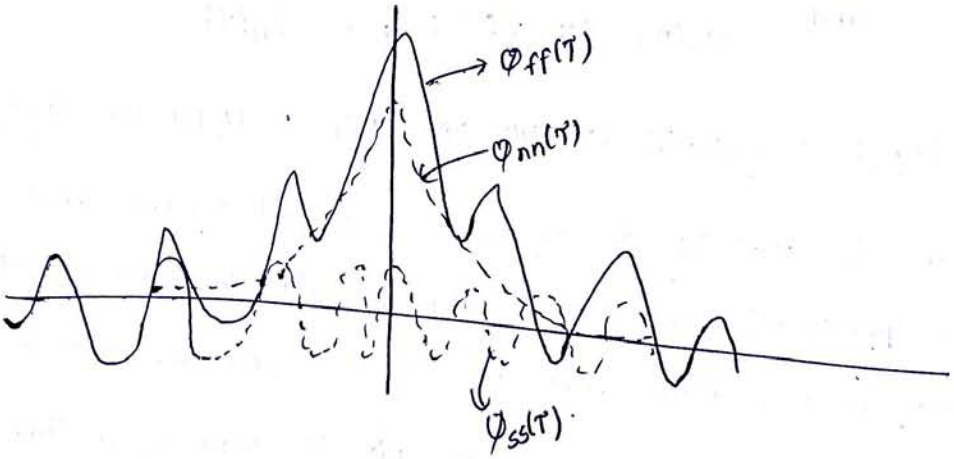
($\because$ $s(t)$ & $n(t)$ are uncorrelated)

$$\bar{\varphi}_{sn}(\tau) = \bar{\varphi}_{ns}(\tau) = 0$$

$$\therefore \quad \bar{\varphi}_{ff}(\tau) = \bar{\varphi}_{ss}(\tau) + \bar{\varphi}_{nn}(\tau).$$

↓

exhibit aperiodic nature at larger values of τ.



→ It follows that $f(t)$ contains a periodic signal of frequency displayed by $\bar{\varphi}_{ff}(\tau)$.

→ If $\bar{\varphi}_{ff}(\tau)$ does exhibit such a periodic nature it is possible to separate $\bar{\varphi}_{ss}(\tau)$ and $\bar{\varphi}_{nn}(\tau)$ {non periodic component}.

EXTRACTION OF SIGNAL FROM NOISE BY FILTERING:

→ A signal masked by noise can be detected either by correlation techniques or filtering.

→ Correlation technique in time domain and filtering in frequency domain.

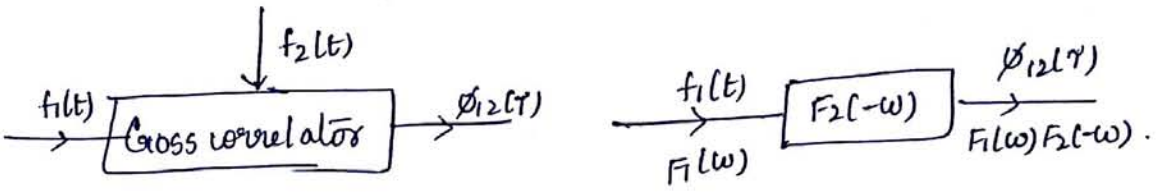


Fig: Crosscorrelation in time & frequency domain.

The impulse response $h(t)$ of a system with a transfer function $F_2(-\omega)$ is given by

$$h(t) = F^{-1}[F_2(-\omega)]$$

But $f_2(t) \longleftrightarrow F_2(\omega)$

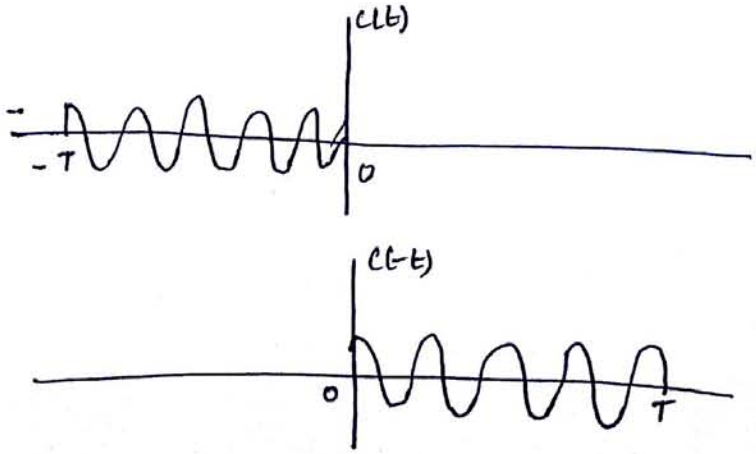
and $f_2(-t) \longleftrightarrow F_2(-\omega)$

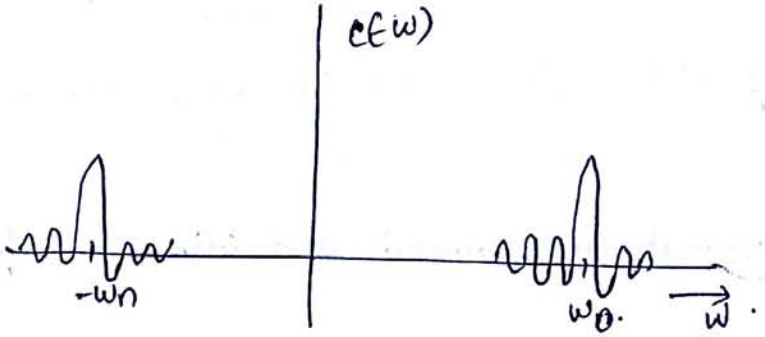
∴ hence $h(t) = f_2(-t)$.

→ The received signal $f(t)$ is

$$f(t) = s(t) + n(t)$$

→ We are filtering out all of the noise signal and extracting the desired periodic signal $s(t)$ by a filter which allows only the frequency components present in $s(t)$ to pass through.





UNIT-5

LAPLACE TRANSFORMS AND Z-TRANSFORMS

LAPLACE TRANSFORMS:

28/10 1, 2, 6, 10, 11, 12, 14, 20, 21, 23, 25, 26, 28, 33, 35, 38, 41, 42, 48, 50, 55,

→ Laplace transform represents continuous time signals in terms of complex exponentials i.e. e^{-st}. It is used to analyze the signals or functions which are not absolutely integrable.

→ More effectively continuous time signals can be analyzed using Laplace transform.

→ Laplace transform provides broader characterization compared to Fourier Transform

DEFINITION:

→ To transform a time domain signal $x(t)$ to S-domain, multiply the signal by e^{-st} and then integrate from $-\infty$ to ∞.

→ The transformed signal is represented as $X(s)$ and transformation is denoted by letter $\mathcal{L}$

Laplace transform is given as for continuous time signal $x(t)$

i.e

$$\boxed{X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} \, dt} \rightarrow (1)$$

where 's' is complex in nature and given as $s = \sigma + j\omega$.

σ → real part / attenuation constant

ω → imaginary / complex frequency

→ If $x(t)$ is defined for $t \geq 0$ (i.e $x(t)$ is causal) then

$$\boxed{\mathcal{L}\{x(t)\} = X(s) = \int_{0}^{\infty} x(t) e^{-st} \, dt} \rightarrow (2)$$

TYPES OF LAPLACE TRANSFORM:

(i) Bilateral or two sided Laplace transform: If the integration is taken from $-\infty$ to ∞ as shown in eq(1) then it is called Bilateral L.T

(ii) Unilateral or one sided Laplace transform: If the integration is taken from 0 to ∞ as shown in eq(2) then it is called Unilateral LT

→ Useful in analysis of networks and solving differential equations

INVERSE LAPLACE TRANSFORM:

→ The s-domain signal $X(s)$ can be transformed to time domain signal $x(t)$ by using inverse laplace transform and is defined as

$$\boxed{L^{-1}[X(s)] = x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds}$$

→ The signal $x(t)$ and $X(s)$ are called Laplace transform pair

$$x(t) \underset{L^{-1}}{\overset{L}{\rightleftarrows}} X(s)$$

RELATION BETWEEN FOURIER TRANSFORM AND LAPLACE TRANSFORM:

Fourier transform is given as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt \rightarrow (1)$$

→ FT can be calculated only if $x(t)$ is absolutely integrable i.e

$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty \rightarrow (2)$$

→ Laplace transform is written as

$$X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-(\sigma+j\omega)t}\,dt \quad \text{putting } s = \sigma + j\omega$$

$$= \int_{-\infty}^{\infty} x(t)\,e^{-\sigma t}\cdot e^{-j\omega t}\,dt$$

$$= \left\{\int_{-\infty}^{\infty} x(t)\,e^{-\sigma t}\right\} e^{-j\omega t}\,dt \rightarrow (3)$$

comparing eq(3) with eq(1), laplace transform of $x(t)$ is basically fourier transform of $x(t)e^{-\sigma t}$.

→ If $\sigma = 0$, then above equation i.e $s = j\omega$ the above eqn

$$X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt = X(j\omega) \text{ when } s = j\omega.$$

→ It is basically fourier transform on imaginary $(j\omega)$ axis in s-plane.

CONVERGENCE / REGION OF CONVERGENCE (ROC):

→ From eqn $\left\{ \int_{-\infty}^{\infty} (x(t)e^{-\sigma t})\, e^{-j\omega t}\, dt \right\}$ we know that laplace transform is basically the fourier transform of $x(t)e^{-\sigma t}$. If fourier transform of $x(t)e^{-\sigma t}$ exists then laplace transform of $x(t)$ exists.

→ $\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}|\, dt < \infty$ must be absolutely integrable for fourier transform to exists.

→ Laplace transform of $x(t)$ will exist, if above condition is satisfied.

→ The range of values of 'σ' for which laplace transform converges is called ROC or region of convergence.

(or)

→ The Laplace transform of a signal given by $\int_{-\infty}^{\infty} x(t)e^{-st}\, dt$. The values of 's' for which the integral $\int_{-\infty}^{\infty} x(t)e^{-st}\, dt$ converges is called ROC.

PROBLEMS:

(1) Calculate the Laplace transform and ROC for $x(t) = e^{-at}u(t)$ (Right sided causal sgl)

Sol

$$x(t) = e^{-at}u(t) \quad \text{where } a>0$$
$$= e^{-at} \quad \text{for } t \geq 0$$

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} e^{-at}\cdot u(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-at} e^{-st}\, dt = \int_{0}^{\infty} e^{-(s+a)t}\, dt$$

$$= \frac{1}{-(s+a)}\left[e^{-(s+a)t}\right]_0^{\infty} = \frac{-1}{s+a}\left[0-1\right]$$

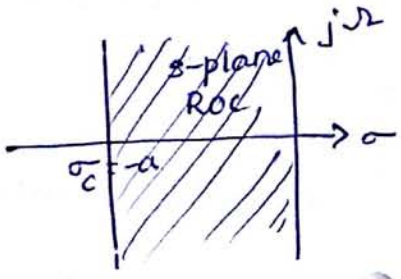
$$= \frac{1}{s+a}$$

ROC: $s > -a$

$$= -\left[\frac{e^{-(s+a)t}}{s+a}\right]_0^{\infty}$$

$$= -\left\{\lim_{t\to\infty}\left[\frac{e^{-(s+a)t}}{s+a}\right] - \lim_{t\to 0}\left[\frac{e^{-(s+a)t}}{s+a}\right]\right\}$$

$$X(s) = +\left[\frac{e^{-(s+a)\infty}}{-(s+a)} - \frac{e^{-(s+a)0}}{-(s+a)}\right]$$



$s = j\omega + \sigma$

$$L\{x(t)\} = \frac{e^{-k\times\infty}e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a}$$

where $k = \sigma + a = \sigma - (-a)$.

when $\sigma > a$, $k = \sigma - (-a) =$ positive $\therefore e^{-k\infty} = e^{-\infty} = 0$

When $\sigma < -a$, $k = \sigma - (-a) =$ Negative $\therefore e^{-k\infty} = e^{+\infty} = \infty$

$\therefore$ $X(s)$ converges when $\sigma > -a$ and does not converge for $\sigma < -a$.

$\therefore$ When $\sigma > -a$, the $X(s)$ is given by

$$L\{x(t)\} = X(s) = \frac{-e^{-k\times\infty}e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} = \frac{-0\times e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} = \frac{1}{s+a}$$

Properties of L.T :

(1) Amplitude Scaling

If $L[x(t)] = X(s)$ then $L[A\,x(t)] = A\,X(s)$

Proof:

$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}\,dt$$

$$L\{A\,x(t)\} = \int_{-\infty}^{\infty} A\cdot x(t)e^{-st}\,dt = A\int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$= A\,X(s).$$

(2) Linearity:

If $L\{x_1(t)\} = X_1(s)$ and $L\{x_2(t)\} = X_2(s)$ then $L\{a_1x_1(t) + a_2x_2(t)\} = a_1X(s) + a_2X_2(s)$

Proof

$$X_1(s) = L\{x_1(t)\} = \int_{-\infty}^{\infty} x_1(t)e^{-st}\,dt$$

$$X_2(s) = L\{x_2(t)\} = \int_{-\infty}^{\infty} x_2(t)e^{-st}\,dt$$

$$L\{a_1x_1(t) + a_2x_2(t)\} = \int_{-\infty}^{\infty}[a_1x_1(t) + a_2x_2(t)]e^{-st}\,dt$$

$$= a_1\int_{-\infty}^{\infty} x_1(t)e^{-st}\,dt + a_2\int_{-\infty}^{\infty} x_2(t)e^{-st}\,dt$$

$$= a_1X_1(s) + a_2X_2(s)$$

3) Time differentiation:

If $L\{x(t)\} = X(s)$ then $L\left\{\frac{d}{dt}x(t)\right\} = sX(s) - X(0)$; where $x(0)$ is value of $x(t)$ at $t=0$

Proof

$$X(s) = L\{x(t)\} = \int_0^{\infty} x(t)e^{-st}\,dt$$

$$\therefore L\left\{\frac{d}{dt}x(t)\right\} = \int_0^{\infty}\frac{dx(t)}{dt}e^{-st} = \int_0^{\infty} e^{-st}\frac{dx(t)}{dt}\cdot dt$$

$\left\{\because \int uv = u\int v - \int\left[du\int v\right]\right.$

$u = e^{-st} \mid v = \frac{dx(t)}{dt}$

$$= \left[e^{-st}\cdot x(t)\right]_0^{\infty} - \int_0^{\infty} -s\,e^{-st}x(t)\,dt$$

$$= e^{-\infty}x(\infty) - e^{0}x(0) + s\int_0^{\infty} x(t)e^{-st}\,dt$$

$$= s\int_0^{\infty} x(t)e^{-st}\,dt - x(0) = sX(s) - x(0)$$

(4) Time Integration:

If $L[x(t)] = X(s)$ then $L\{\int x(t)\,dt\} = \frac{X(s)}{s} + \frac{\left[\int x(t)\,dt\right]_{t=0}}{s}$

Proof

$$X(s) = L\{x(t)\} = \int_0^\infty x(t)\,e^{-st}\,dt$$

$$L\{\int x(t)\,dt\} = \int_0^\infty \left[\int x(t)\,dt\right] e^{-st}\,dt$$

$$= \left[\left[\int x(t)\,dt\right]\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty x(t)\cdot\frac{e^{-st}}{-s}\,dt$$

$$= \left[\int x(t)\,dt\right]\Big|_{t=\infty}\frac{e^{-\infty}}{-s} - \left[\int x(t)\,dt\right]\Big|_{t=0}\frac{e^{0}}{-s} + \frac{1}{s}\int_0^\infty x(t)\,e^{-st}\,dt$$

$$= \frac{1}{s}\left[\int x(t)\,dt\right]\Big|_{t=0} + \frac{1}{s}\int_0^\infty x(t)\,e^{-st}\,dt$$

$$= \frac{X(s)}{s} + \frac{\left[\int x(t)\,dt\right]\Big|_{t=0}}{s}$$

5) Frequency shifting:

If $L[x(t)] = X(s)$ then $L\{e^{\pm at}x(t)\} = X(s \mp a)$

Proof

$$X(s) = L\{x(t)\} = \int_{-\infty}^\infty x(t)\,e^{-st}\,dt$$

$$\therefore L\{e^{\pm at}x(t)\} = \int_{-\infty}^\infty e^{\pm at}x(t)\cdot e^{-st}\,dt$$

$$= \int_{-\infty}^\infty x(t)\cdot e^{-(s\mp a)t}\,dt$$

$$= X(s \mp a).$$

6) Time shifting

If $L\{x(t)\} = X(s)$ then $L\{x(t \pm a)\} = e^{\pm as}X(s)$

Proof:

$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$L\{x(t \pm a)\} = \int_{-\infty}^{\infty} x(t \pm a) e^{-st} dt = \int_{-\infty}^{\infty} x(\tau) e^{-s(\tau \mp a)} d\tau$$

put $t \pm a = \tau$
$\therefore t = \tau \mp a$
$dt = d\tau$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-s\tau} \times e^{\pm as} d\tau = e^{\pm as} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau} d\tau$$

$$= e^{\pm as} \int_{-\infty}^{\infty} x(t) e^{-st} dt = e^{\pm as} (X(s))$$

(7) Frequency differentiation:

If $L\{x(t)\} = X(s)$ then $L\{t x(t)\} = -\frac{d}{ds} X(s)$

Proof

$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\frac{d}{ds} X(s) = \frac{d}{ds} \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x(t) \left(\frac{d}{ds} e^{-st}\right) dt = \int_{-\infty}^{\infty} x(t) (-t e^{-st}) dt$$

$$= \int_{-\infty}^{\infty} (-t x(t)) e^{-st} dt = L\{-t x(t)\} = -L\{t x(t)\}$$

$$\therefore L\{t x(t)\} = -\frac{d}{ds} X(s).$$

(8) Frequency Integration:

If $L\{x(t)\} = X(s)$ then $L\left\{\frac{1}{t} x(t)\right\} = \int_{s}^{\infty} X(s)\, ds$.

Proof

$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

On integrating above eqn w.r.to s b/w limits s to ∞

$$\int_{s}^{\infty} X(s)\, ds = \int_{s}^{\infty} \left[\int_{-\infty}^{\infty} x(t) e^{-st} dt\right] ds = \int_{-\infty}^{\infty} x(t) \left[\int_{s}^{\infty} e^{-st} ds\right] dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[\frac{e^{-st}}{-t}\right]_{s}^{\infty} dt = \int_{-\infty}^{\infty} x(t) \left[\frac{e^{\infty}}{-t} - \frac{e^{-st}}{-t}\right] dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[0 + \frac{e^{-st}}{t}\right] dt = \int_{-\infty}^{\infty} \left[\frac{1}{t} x(t)\right] e^{-st} dt = L\left\{\frac{1}{t} x(t)\right\}.$$

9) Time Scaling:

If $L[x(t)] = X(s)$ then $L\{x(at)\} = \frac{1}{|a|} X\left(\frac{s}{a}\right)$

Proof $X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

$$\therefore L\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-st} dt = \int_{-\infty}^{\infty} x(\tau) e^{-s\left(\frac{\tau}{a}\right)} \frac{d\tau}{a} \quad \left\{ \begin{array}{l} \text{put } at = \tau \\ \therefore dt = \frac{d\tau}{a} \end{array} \right.$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} X(\tau) e^{-\left(\frac{s}{a}\right)\tau} d\tau$$

$$= \frac{1}{a} X\left(\frac{s}{a}\right)$$

The above transform is applicable for positive values of 'a'.

If 'a' happens to be negative

$$L\{x(at)\} = \frac{-1}{a} X\left(\frac{s}{a}\right)$$

In general, $L\{x(at)\} = \frac{1}{|a|} X\left(\frac{s}{a}\right)$.

(10) Periodicity:

If $x(t) = x(t+nT)$ and $x_1(t)$ be one period of $x(t)$ and $L\{x_1(t)\} = \int_0^T x_1(t) e^{-st} dt$

then $L\{x(t+nT)\} = \frac{1}{1-e^{-sT}} \int_0^T x_1(t) e^{-st} dt$

Proof

$$L\{x(t+nT)\} = \int_0^{\infty} x(t+nT) e^{-st} dt$$

$$= \int_0^T x_1(t) e^{-st} dt + \int_T^{2T} x_1(t-T) e^{-s(t+T)} dt + \int_{2T}^{3T} x_1(t-2T) e^{-s(t+2T)} dt + \cdots$$

$$+ \cdots + \int_{pT}^{(p+1)T} x_1(t-pT) e^{-s(t+pT)} dt + \cdots$$

$$= \sum_{p=0}^{\infty} \int_{pT}^{(p+1)T} x_1(t-pT) e^{-s(t+pT)} dt$$

$$= \sum_{p=0}^{\infty} \int_0^T x_1(t) e^{-st} \cdot e^{-pst} dt = \int_0^T x_1(t) e^{-st} \left(\sum_{p=0}^{\infty} e^{-pst}\right) dt$$

$$= \int_0^T x_1(t) e^{-st} \left(\sum_{p=0}^{\infty} e^{-sT}\right)^p dt$$

$$= \int_0^T x_1(t) e^{-st} \left(\frac{1}{1-e^{-sT}}\right) dt = \frac{1}{1-e^{-sT}} \int_0^T x_1(t) e^{-st} dt.$$

INVERSE LAPLACE TRANSFORM: (PARTIAL FRACTION EXPANSION)

The inverse laplace transform by partial fraction method of all three cases.

Case i) when s-domain signal $X(s)$ has distinct poles

Let $X(s) = \dfrac{K}{s(s+P_1)(s+P_2)}$

By partial fraction

$$X(s) = \frac{k_1}{s} + \frac{k_2}{s+P_1} + \frac{k_3}{s+P_2}$$

The residues k_1, k_2, k_3 are given by

$$k_1 = X(s) \times s \Big|_{s=0}, \quad k_2 = X(s) \times (s+P_1) \Big|_{s=-P_1}$$

$$k_3 = X(s) \times (s+P_2) \Big|_{s=-P_2}.$$

$$L^{-1}\{X(s)\} = L^{-1}\left\{\frac{k_1}{s} + \frac{k_2}{s+P_1} + \frac{k_3}{s+P_2}\right\}$$

$$x(t) = k_1 L^{-1}\left\{\frac{1}{s}\right\} + k_2 L^{-1}\left\{\frac{1}{s+P_1}\right\} + k_3 L^{-1}\left\{\frac{1}{s+P_2}\right\}$$

$$= k_1 u(t) + k_2 e^{-P_1(t)} u(t) + k_3 e^{-P_2 t} u(t).$$

Case ii) when s-domain signal $X(s)$ has multiple poles

$$X(s) = \frac{k}{s(s+P_1)(s+P_2)^r}$$

$$X(s) = \frac{k_1}{s} + \frac{k_2}{s+P_1} + \frac{k_3}{(s+P_2)^r} + \frac{k_4}{s+P_2}$$

The residues k_1, k_2, k_3, k_4 are given by

$$k_1 = X(s) \times s \Big|_{s=0} \qquad k_2 = X(s) \times (s+P_1) \Big|_{s=-P_1}$$

$$k_3 = X(s) \times (s+P_2)^r \Big|_{s=-P^r} \qquad k_4 = \frac{d}{ds}\left[X(s) \times (s+P_2)^r\right]_{s=-P2}$$

$$L^{-1}\{X(s)\} = L^{-1}\left\{\frac{k_1}{s} + \frac{k_2}{s+P_1} + \frac{k_3}{(s+P_2)^r} + \frac{k_4}{s+P_2}\right\}$$

$$= k_1 u(t) + k_2 e^{-P_1 t} u(t) + k_3 t e^{-P_2 t} u(t) + k_4 e^{-P_2 t} u(t)$$

In general

$$X(s) = \frac{k}{s(s+P_1)(s+P_2)^q} \quad \text{then}$$

$$X(s) = \frac{k_1}{s} + \frac{k_2}{s+P_1} + \frac{k_3}{(s+P_2)^q} + \frac{k_4}{(s+P_2)^{q-1}} + \dots + \frac{k_{(q-1)2}}{s+P_2}$$

$$k_{r2} = \frac{1}{r!} \frac{d^r}{dq^r}\left[X(s) \times (s+2)^q \right] ; r = 1, 2 \dots q-1.$$

Case iii) When s-domain signal $X(s)$ has complex conjugate poles

$$\text{Let } X(s) = \frac{k}{(s+P_1)(s^2+bs+c}$$

$$X(s) = \frac{k_1}{s+P_1} + \frac{k_2 s + k_3}{s^2+bs+c}$$

$$k_1 = X(s) \times (s+P_1)_{s=-P_1}$$

$$X(s) = \frac{k_1}{s+P_1} + \frac{k_2 s + k_3}{s^2 + 2 \times \frac{b}{2} \cdot s + \left(\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2}$$

Arranging s^2+bs in $(x+y)^2$.

$$= \frac{k_1}{s+P_1} + \frac{k_2 s + k_3}{\left(s+\frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)^2}$$

put $\frac{b}{2} = a$, $c - \frac{b^2}{4} = \Omega_0^2$

$$X(s) = \frac{k_1}{s+P_1} + k_2 \cdot \frac{s + a + \frac{k_3}{k_2} - a}{(s+a)^2 + \Omega_0^2}$$

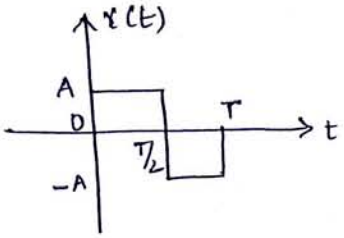
$$= \frac{k_1}{s+P_1} + k_2 \cdot \frac{s+a+k_4}{(s+a)^2+\Omega_0} \quad \left(\because \text{put } \frac{k_3}{k_2} - a = k_4 \right)$$

$$\therefore X(s) = \frac{k_1}{s+P_1} + k_2 \cdot \frac{s+a}{(s+a)^2+\Omega_0^2} + k_5 \frac{\Omega_0}{(s+a)^2+\Omega_0^2} \quad \left(\frac{k_2 k_4}{\Omega_0} = k_5 \right).$$

$$x(t) = k_1 e^{-P_1 t} u(t) + k_2 e^{-at} \cos \Omega_0 t \, u(t) + k_5 e^{-at} \sin \Omega_0 t \, u(t).$$

LAPLACE TRANSFORM OF CERTAIN SIGNALS USING WAVEFORM SYNTHESIS:

(a)



proof

$x(t) = A$ for $0 < t < T/2$

$= -A$ for $T/2 < t < T$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{T} x(t)\, e^{-st}\, dt$$

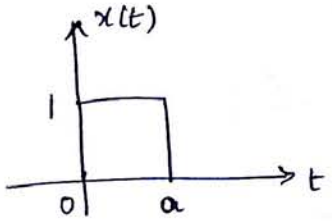
$$= \int_{0}^{T/2} A e^{-st}\, dt + \int_{T/2}^{T} (-A) e^{-st}\, dt = \left[\frac{A e^{-st}}{-s}\right]_{0}^{T/2} + \left[\frac{-A e^{-st}}{-s}\right]_{T/2}^{T}$$

$$= \left[\frac{A e^{-sT/2}}{-s} - \frac{A e^{0}}{-s}\right] + \left[\frac{A e^{-sT}}{s} - \frac{A e^{-sT/2}}{s}\right]$$

$$= -\frac{A e^{-\frac{sT}{2}}}{s} + \frac{A}{s} + \frac{A e^{-sT}}{s} - \frac{A e^{-sT/2}}{s}$$

$$= \frac{A}{s}\left[1 - e^{-sT/2}\right]$$

(b)



proof

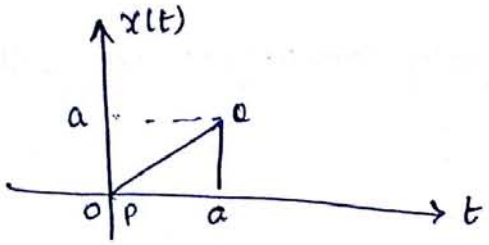
$x(t) = 1$ for $0 \le t \le a$

$= 0$ for $t > a$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{a} 1 \times e^{-st}\, dt = \int_{0}^{a} e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_{0}^{a} = \frac{e^{-as}}{-s} - \frac{e^{0}}{-s} = \frac{-e^{-as}}{s} + \frac{1}{s} = \frac{1}{s}(1 - e^{-as})$$

(C)



Consider the eqn of straight line $\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}$.

Here $y = x(t)$, $x = t$

Consider points P and Q as shown in figure

$$P = [0, 0], \quad Q\,[a, a]$$
$$t_1, x_1(t) \qquad t_2, x_2(t)$$

$$\therefore \frac{x(t)-0}{0-a} = \frac{t-0}{0-a} \Rightarrow x(t) = t$$

$$\therefore x(t) = t \quad \text{for } t = 0 \text{ to } a$$
$$= 0 \quad \text{for } t > a$$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt = \int_0^a e^{-st} \cdot t\, dt$$

$$= \left[t \times \frac{e^{-st}}{-s} - \int 1 \cdot \frac{e^{-st}}{-s} dt \right]_0^a$$

$$= \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^a$$

$$= \left[-\frac{ae^{-sa}}{s} - \frac{e^{-sa}}{s^2} + 0 + \frac{e^0}{s^2} \right]$$

$$= \frac{1}{s^2} - \frac{e^{-as}}{s^2} - \frac{ae^{-as}}{s}$$

$$= \frac{1}{s^2}\left[1 - e^{-as}(1 + as) \right]$$

Problems on Laplace transforms:

(1) Determine the Laplace transform of continuous time signals and their ROC

(a) $x(t) = A\,u(t)$

$x(t) = A$ for $t \geq 0$ bcoz $u(t) = 1$ for $t \geq 0$
$= 0$ for $t < 0$

Laplace transform

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{\infty} A \cdot e^{-st}\, dt = A\left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= A\left[\frac{e^{-(\sigma+j\omega)t}}{-s}\right]_0^{\infty}$$

$$= A\left[\frac{e^{-(\sigma+j\omega)\times\infty}}{-s} + \frac{e^0}{s}\right] = A\left[\frac{e^{-\sigma\times\infty} \cdot e^{-j\omega\times\infty}}{-s} + \frac{e^0}{s}\right]$$

when $\sigma > 0$; $e^{-\sigma\times\infty} = e^{-\sigma\infty} = e^{-\infty} = 0$

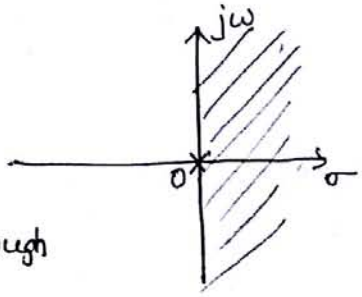
when $\sigma < 0$; $e^{-\sigma\times\infty} = e^{\infty} = \infty$

$\therefore$ we can say that $X(s)$ converges when $\sigma > 0$

When $\sigma > 0$, the $X(s)$ is given by

$$X(s) = A\left[\frac{0 \times e^{-j\omega\times\infty}}{-s} + \frac{1}{s}\right] = \frac{A}{s}$$

$\therefore L\{A\,u(t)\} = \frac{A}{s}$; with ROC as all point is s-plane to the right of line passing through $\sigma = 0$



(or ROC is right half of s-plane).

(2) $x(t) = t\,u(t)$

Sol

$$x(t) = t \begin{cases} u(t) = 1 & \text{for } t \geq 0 \\ \quad\;\; = 0 & \text{otherwise} \end{cases}$$

$$\int uv = u\int v - \int\left[du \int v\right]$$

$$L\{x(t)\} = X(s) = \int_0^\infty x(t)\, e^{-st}\, dt$$

$$= \int_0^\infty t \cdot e^{-st}\, dt = t\left[\frac{e^{-st}}{-s}\right]_0^\infty - \int\left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= t\left[\frac{e^{-st}}{-s}\right]_0^\infty - \left[\frac{e^{-st}}{s^2}\right]_0^\infty$$

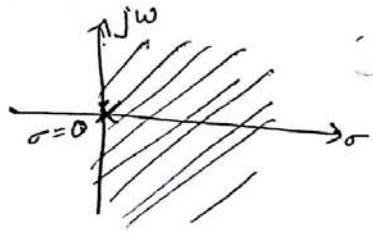
$$= t\left[\frac{e^{-(\sigma+j\omega)t}}{-s}\right]_0^\infty - \left[\frac{e^{-(\sigma+j\omega)t}}{s^2}\right]_0^\infty$$

$$= \left[\infty \times \frac{e^{-(\sigma+j\omega)\infty}}{-s} - 0 - \frac{e^{-(\sigma+j\omega)\infty}}{s^2} + \frac{e^0}{s^2}\right]$$

$$= \left[\infty \times \frac{e^{-\sigma\times\infty} \cdot e^{-j\omega\times\infty}}{-s} - \frac{e^{-\sigma\times\infty} \cdot e^{-j\omega\times\infty}}{s^2} + \frac{1}{s^2}\right]$$

when $\sigma > 0$, positive i.e $e^{-\sigma\times\infty} = e^{-\sigma\infty} = 0$

when $\sigma < 0$, negative i.e $e^{-\sigma\times\infty} = e^{\sigma\infty} = \infty$



It converges when $\sigma > 0$

$$= \left[\infty \times 0 \times e^{-j\omega\times\infty} - \frac{0 \times e^{-j\omega\times\infty}}{s^2} + \frac{1}{s^2}\right] = \frac{1}{s^2}$$

$\therefore$ ROC is right half of s-plane.

3) Given that $x(t) = e^{-3t}u(t)$:

Sol $x(t) = e^{-3t}$ for $t \geq 0$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st}\, dt$$

$$= \int_0^\infty x(t)\cdot e^{-st}\,dt = \int_0^\infty e^{-3t}\cdot e^{-st}\,dt = \int_0^\infty e^{-(s+3)t}\,dt.$$

$$= \left[\frac{e^{-(s+3)t}}{-(s+3)}\right]_0^\infty$$

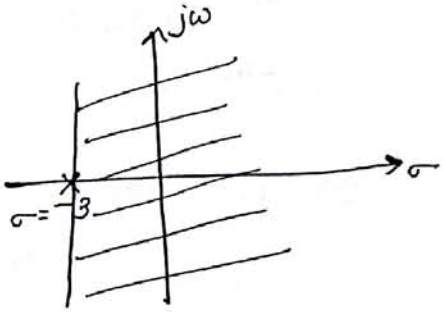
$$= \left[\frac{e^{-(s+3)\infty}}{-(s+3)} + \frac{e^{-(s+3)\cdot 0}}{s+3}\right] \Rightarrow \frac{e^{-(\sigma+j\omega+3)\times\infty}}{-(s+3)} + \frac{1}{s+3}$$

$$= \left[\frac{e^{-(\sigma+3)\infty}\cdot e^{-j\omega\times\infty}}{-(s+3)} + \frac{1}{s+3}\right]$$

$\sigma+3 = \sigma-(-3)$

if $\sigma > -3$ = positive $\therefore e^{-k\times\infty} = e^{-k\infty} = 0$

if $\sigma < -3$ = negative $\therefore e^{-k\times\infty} = e^{k\infty} = \infty$



$\therefore$ It converges at $\sigma > -3$.

$$x(s) = \left[\frac{0\times e^{-j\omega\times\infty}}{-(s+3)} + \frac{1}{s+3}\right] = \frac{1}{s+3}$$

$L\{e^{-3t}u(t)\} = \frac{1}{s+3}$; with ROC as all pts in s-plane to right of line passing through $\sigma=-3$

(4) $x(t) = e^{-3t}u(-t)$

Sol $x(t) = e^{-3t}$ for $t \leq 0$

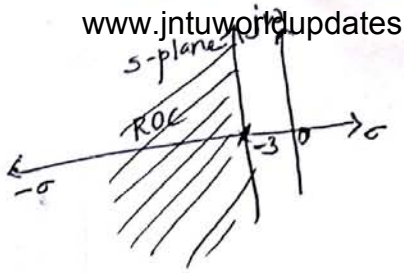
$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt = \int_{-\infty}^{0} e^{-3t}\cdot e^{-st}\,dt$$

$$= \int_{-\infty}^{0} e^{-(s+3)t}\,dt$$

$$= \left.\frac{e^{-(s+3)t}}{-(s+3)}\right]_{-\infty}^{0} = \frac{e^0}{-(s+3)} + \frac{e^{+(s+3)\times\infty}}{+(s+3)}$$

$$= \frac{-1}{s+3} + \frac{e^{+(\sigma+j\omega+3)\times\infty}}{s+3}$$

$$= \frac{-1}{s+3} + \frac{e^{+(\sigma+3)\times\infty} \cdot e^{-j\omega\times\infty}}{s+3}$$



$k = \sigma + 3 = \sigma - (-3)$

when $\sigma > -3$; positive $e^{+(\sigma+3)\times\infty} = \infty$

when $\sigma < -3$; negative $e^{-k\infty} = e^{k\infty} = 0$

$\therefore$ ROC it converges when $\sigma < -3$ $\Rightarrow \frac{-1}{s+3} + \frac{0 \times e^{-j\omega\times\infty}}{s+3}$

$L\{x(t)\} = L\{e^{-3t}u(-t)\} = \frac{-1}{s+3}$ { ROC as all pts in splane to the left passing through $\sigma = -3$

---

(5) $x(t) = e^{-4|t|}$

$= e^{-4t}$ for $t \geq 0$

$= e^{+4t}$ for $t \leq 0$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt = \int_{-\infty}^{0} e^{4t} \cdot e^{-st} dt + \int_{0}^{\infty} e^{-4t} \cdot e^{-st} dt$$

$$= \int_{-\infty}^{0} e^{-(s-4)t} dt + \int_{0}^{\infty} e^{-(s+4)t} dt$$

$$= \left[\frac{e^{-(s-4)t}}{-(s-4)}\right]_{-\infty}^{0} + \left[\frac{e^{-(s+4)t}}{-(s+4)}\right]_{0}^{\infty}$$

$$= \left[\frac{1}{-(s-4)} + \frac{e^{+(s-4)\times\infty}}{s-4} - \frac{e^{-(s+4)\times\infty}}{s+4} + \frac{1}{s+4}\right]$$

$$= \left[\frac{-1}{s-4} + \frac{e^{(\sigma+j\omega-4)\times\infty}}{s-4} - \frac{e^{-(\sigma+j\omega+4)\times\infty}}{s+4} + \frac{1}{s+4}\right]$$

$$= \left[\frac{-1}{s-4} + \frac{e^{(\sigma-4)\times\infty} \cdot e^{+j\omega\times\infty}}{s-4} - \frac{e^{-(\sigma+4)\times\infty} \cdot e^{-j\omega\times\infty}}{s+4} + \frac{1}{s+4}\right]$$

$\sigma - 4 \Rightarrow \sigma - (+4)$

when $\sigma > 4$ positive $e^{k\times\infty} = \infty$

when $\sigma < 4$ negative $e^{-k\infty} = 0$

It converges when $\sigma < +4$

$\sigma + 4 \rightarrow \sigma - (-4)$

when $\sigma \geq 4$ positive $e^{-k\infty} = 0$

when $\sigma < -4$ negative $e^{+k\infty} = \infty$

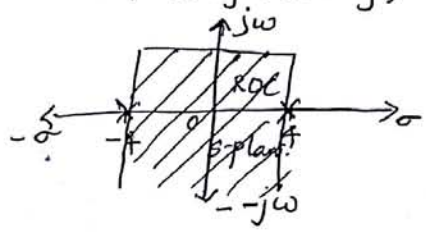
It converges when $\sigma > -4$

when σ lies between -4 and $+4$, the $X(s)$ is given by

$$X(s) = \left[\frac{-1}{s-4} + \frac{e^{(\sigma-4)\times\infty} e^{j\omega\times\infty}}{s-4} - \frac{e^{-(\sigma+4)\times\infty} \cdot e^{-j\omega\times\infty}}{s+4} + \frac{1}{s+4} \right]$$

$$= \frac{-1}{s-4} + \frac{1}{s+4} \Rightarrow \frac{-s-4+s-4}{s^2-16} = \frac{-8}{s^2-16}.$$

with ROC as all points in splane in between the lines passing through $\sigma = -4$ and $\sigma = 4$.



(6) Determine the Laplace transform of following signals.

$x(t) = \sin\omega_0 t \, u(t)$

Sol $x(t) = \sin\omega_0 t$ for $t \geq 0$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$\because \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$

$$= \int_{+0}^{\infty} \sin\omega_0 t \cdot e^{-st} dt$$

$$= \int_0^{\infty} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \cdot e^{-st} dt$$

$$= \frac{1}{2j} \left[\int_0^{\infty} e^{-(s-j\omega_0)t} dt - \int_0^{\infty} e^{-(s+j\omega_0)t} dt \right]$$

$$= \frac{1}{2j} \left\{ \left[\frac{e^{-(s-j\omega_0)t}}{-(s-j\omega_0)} \right]_0^{\infty} - \left[\frac{e^{-(s+j\omega_0)t}}{-(s+j\omega_0)} \right]_0^{\infty} \right]$$

$$= \frac{1}{2j} \left[\frac{e^{-\infty}}{-(s-j\omega_0)} + \frac{e^0}{s-j\omega_0} + \frac{e^{-\infty}}{s+j\omega_0} - \frac{e^0}{s+j\omega_0} \right]$$

$$= \frac{1}{2j} \left[\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0} \right] = \frac{1}{2j} \left[\frac{\cancel{s}+j\omega_0 - \cancel{s}+j\omega_0}{s^2 - j^2\omega_0^2} \right] = \frac{1}{\cancel{2j}} \cdot \frac{\cancel{2j}\omega_0}{s^2+\omega_0^2}.$$

$$= \omega_0 / s^2+\omega_0^2.$$

(7) $x(t) = \cos\omega_0 t\, u(t)$

Sol $x(t) = \cos\omega_0 t \quad ; t \geq 0$

$$L\{x(t)\} = X(s) = \int_0^\infty x(t) \cdot e^{-st} dt = \int_0^\infty \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \cdot e^{-st} dt$$

$$= \frac{1}{2} \int_0^\infty \left[e^{-(s-j\omega_0)t} + e^{-(s+j\omega_0)t} \right] dt$$

$$= \frac{1}{2} \left[\frac{e^{-(s-j\omega_0)t}}{-(s-j\omega_0)} + \frac{e^{-(s+j\omega_0)t}}{-(s+j\omega_0)} \right]_0^\infty$$

$$= \frac{1}{2} \left[\frac{e^{-\infty}}{-(s-j\omega_0)} + \frac{e^0}{s-j\omega_0} - \frac{e^{-\infty}}{s+j\omega_0} + \frac{e^0}{s+j\omega_0} \right]$$

$$= \frac{1}{2} \left[\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0} \right] = \frac{1}{2} \left[\frac{s+\cancel{j\omega_0}+s-\cancel{j\omega_0}}{s^2-j^2\omega_0^2} \right] = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}s}{s^2+\omega_0^2}$$

$$L\{\cos\omega_0 t\, u(t)\} = \frac{s}{s^2+\omega_0^2}$$

8) $x(t) = \cosh\omega_0 t\; u(t)$

Sol $x(t) = \cosh\omega_0 t$ for $t \geq 0$ $\qquad \because \cosh\theta = \frac{e^\theta + e^{-\theta}}{2}$

$$L\{x(t)\} = X(s) = \int_{-\infty}^\infty x(t) \cdot e^{-st} dt$$

$$= \int_0^\infty \left(\frac{e^{\omega_0 t} + e^{-\omega_0 t}}{2} \right) e^{-st} dt$$

$$= \frac{1}{2} \int_0^\infty e^{\omega_0 t} \cdot e^{-st} dt + \frac{1}{2} \int_0^\infty e^{-\omega_0 t} \cdot e^{-st} dt$$

$$= \frac{1}{2} \left[\int_0^\infty e^{-(s-\omega_0)t} dt + \int_0^\infty e^{-(s+\omega_0)t} dt \right]$$

$$= \frac{1}{2} \left\{ \left[\frac{e^{-(s-\omega_0)t}}{-(s-\omega_0)} \right]_0^\infty + \left[\frac{e^{-(s+\omega_0)t}}{-(s+\omega_0)} \right]_0^\infty \right.$$

$$= \frac{1}{2} \left[\frac{e^{-\infty}}{-(s-\omega_0)} + \frac{1}{s-\omega_0} + \frac{e^{-\infty}}{-(s+\omega_0)} + \frac{1}{s+\omega_0} \right]$$

$$= \frac{1}{2}\left[\frac{1}{s-\omega_0} + \frac{1}{s+\omega_0}\right]$$

$$= \frac{1}{2}\left[\frac{s+\cancel{\omega_0}+s-\cancel{\omega_0}}{s^2-\omega_0^2}\right] = \frac{s}{s^2-\omega_0^2}.$$

9) $x(t) = e^{-at}\sin\omega_0 t\, u(t)$

Sol $x(t) = e^{-at}\sin\omega_0 t$ for $t \geq 0$

$$L\{x(t)\} = X(s) = \int_0^\infty x(t)\cdot e^{-st}\,dt = \int_0^\infty e^{-at}\sin\omega_0 t\, e^{-st}\,dt = \int_0^\infty e^{-at}\left(\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j}\right)e^{-st}dt$$

$$= \frac{1}{2j}\left[\int_0^\infty e^{-(s+a-j\omega_0)t} - e^{-(s+a+j\omega_0)t}\right]dt$$

$$= \frac{1}{2j}\left[\frac{e^{-(s+a-j\omega_0)t}}{-(s+a-j\omega_0)}\right]_0^\infty - \left[\frac{e^{-(s+a+j\omega_0)t}}{-(s+a+j\omega_0)}\right]_0^\infty$$

$$= \frac{1}{2j}\left[\frac{e^{-\infty}}{-(s+a-j\omega)} + \frac{e^0}{s+a-j\omega_0} + \frac{e^{-\infty}}{s+a+j\omega} - \frac{e^0}{s+a+j\omega_0}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{s+a-j\omega_0} - \frac{1}{s+a+j\omega_0}\right] = \frac{1}{2j}\left[\frac{\cancel{s}+\cancel{a}+j\omega_0-\cancel{s}-\cancel{a}+j\omega_0}{(s+a)^2+\omega_0^2}\right]$$

$$= \frac{\cancel{2j}\omega_0}{\cancel{2j}\cdot(s+a)^2+\omega_0^2}$$

$$= \frac{\omega_0}{(s+a)^2+\omega_0^2}.$$

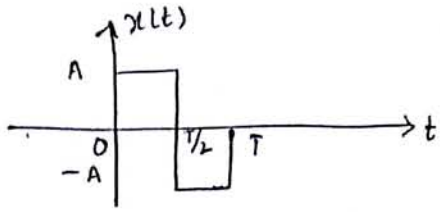
(10) $x(t) = e^{-at}\cos\omega_0 t\, u(t)$

Sol $x(t) = e^{-at}\cos\omega_0 t$ for $t \geq 0$.

$$\text{||}^{ly} = \frac{s+a}{(s+a)^2+\omega_0^2}$$

(11) Determine the Laplace transform of following signals.

Sol)



$x(t) = A$ for $0<t<T/2$

$= -A$ for $T/2<t<T$

$$L\{x(t)\} = X(s) = \int_0^{T/2} A\cdot e^{-st}dt + \int_{T/2}^{T} -A\cdot e^{-st}dt$$

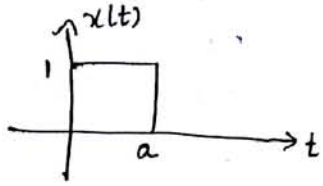
$$= A\left[\frac{e^{-st}}{-s}\right]_0^{T/2} - A\left[\frac{e^{-st}}{-s}\right]_{T/2}^{T}$$

$$= A\left[\frac{e^{-sT/2}}{-s} + \frac{1}{s}\right] - A\left[\frac{e^{-sT}}{-s} + \frac{e^{-sT/2}}{s}\right]$$

$$= -\frac{Ae^{-sT/2}}{s} + \frac{A}{s} + \frac{Ae^{-sT}}{s} - \frac{Ae^{-sT/2}}{s}$$

$$= \frac{A}{s}\left[1 + e^{-sT} - 2e^{-sT/2}\right] = \frac{A}{s}\left[1 - e^{-sT/2}\right]$$

(12)



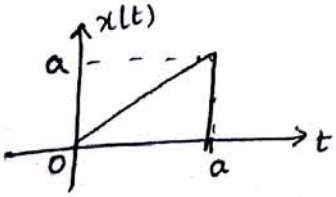
$x(t) = 1$ for $0<t<a$

$= 0$ otherwise

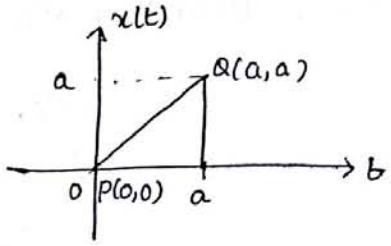
$$L\{x(t)\} = X(s) = \int_0^a e^{-st}dt = \left.\frac{e^{-st}}{-s}\right]_0^a$$

$$= \frac{e^{-as}}{-s} + \frac{1}{s} \Rightarrow \frac{1}{s}\left[1 - e^{-as}\right]$$

(13)



$$\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}.$$



here $(x_1,y_1) = (0,0)$, $(x_2,y_2) = (a,a)$

$x = t$, $y = x(t)$.

$$\Rightarrow \frac{x(t)-0}{a-a} = \frac{t-0}{0-a} \Rightarrow \frac{x(t)}{-a} = \frac{t}{-a} \Rightarrow x(t) = t \text{ for } 0<t<a$$
$$= 0 \text{ otherwise}$$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t).e^{-st}\,dt$$

$$= \int_0^a t e^{-st}\,dt = \left\{ t\left[\frac{e^{-st}}{-s}\right]_0^a - \int \frac{e^{-st}}{-s}\Big]_0^a \right\}$$

$$= t.\frac{e^{-st}}{-s}\Big]_0^a - \frac{e^{-st}}{s^2}\Big]_0^a$$

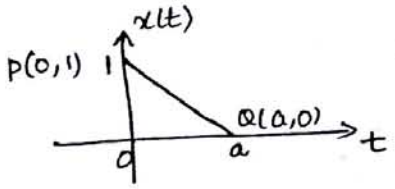
$$= a.\frac{e^{-sa}}{-s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2}$$

$$= \frac{1}{s^2}\left[1 - e^{-as} - s.e^{-as}.a\right]$$

$$= \frac{1}{s^2}\left[1 - e^{-as} - as.e^{-as}\right]$$

$$= \frac{1}{s^2}\left[1 - e^{-as}(1+as)\right]$$

Q14)



$(x_1,y_1) = (0,1)$, $(x_2,y_2) = (a,0)$, $x = t$, $y = x(t)$.

$$\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2} \Rightarrow \frac{x(t)-1}{1-0} = \frac{t-0}{0-a} \Rightarrow x(t)-1 = \frac{-t}{a}$$

$x(t) = 1 - t/a$ for 0 to a

$= 0$ for $t > a$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}\,dt$$

$$= \int_0^a \left(1-\frac{t}{a}\right) e^{-st}\,dt = \int_0^a e^{-st}\,dt - \int_0^a \frac{t}{a} e^{-st}\,dt$$

$$= \left.\frac{e^{-st}}{-s}\right]_0^a - \frac{1}{a}\left[\int_0^a t\cdot e^{-st}\,dt\right]$$

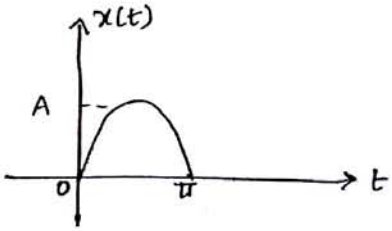
$$= \frac{e^{-as}}{-s} + \frac{1}{s} - \frac{1}{a}\left[t\cdot\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^a$$

$$= \frac{e^{-as}}{-s} + \frac{1}{s} - \frac{1}{\cancel{a}}\cdot\cancel{a}\cdot\frac{e^{-as}}{-s} + \frac{1e^{-sa}}{as^2} - \frac{1}{as^2}$$

$$= \cancel{\frac{e^{-as}}{-s}} + \frac{1}{s} + \cancel{\frac{e^{-as}}{s}} + \frac{e^{-as}}{as^2} - \frac{1}{as^2}$$

$$= \frac{1}{s} + \frac{e^{-as}}{as^2} - \frac{1}{as^2} = \frac{1}{as^2}\left[e^{-as} + as - 1\right]$$

(15) Determine the laplace transform of sine pulse



Sol $x(t) = A\sin t$ for $0<t<\pi$

$= 0$ for $t>\pi$

$$L\{x(t)\} = X(s) = \int_0^{\pi} A\cdot\sin t\, e^{-st}\,dt$$

$$= A\left[\frac{e^{jt}-e^{-jt}}{2j}\, e^{-st}\right]dt = \frac{A}{2j}\left[\int_0^{\pi} e^{-(s-j)t}\,dt - \int_0^{\pi} e^{-(s+j)t}\,dt\right]_0$$

$$= \frac{A}{2j}\left[\frac{e^{-(s-j)t}}{-(s-j)}\right]_0^{\pi} - \frac{A}{2j}\left[\frac{e^{-(s+j)t}}{-(s+j)}\right]_0^{\pi}$$

$$= \frac{A}{2j}\left[\frac{e^{-(s+j)t}}{s+j} - \frac{e^{-(s-j)t}}{s-j}\right]_0^{\pi} = \frac{A}{2j}\left[\frac{(s-j)e^{-st}\cdot e^{-jt} - (s+j)e^{-st}e^{+jt}}{s^2-j^2}\right]_0^{\pi}$$

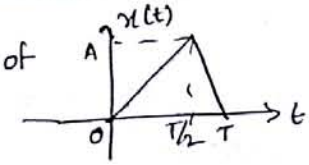
$$= \frac{A}{2j(s^2+1)}\left[(s-j)e^{-st}e^{-jt} - (s+j)e^{-st}e^{jt}\right]_0^{\pi}$$

$$= \frac{A}{2j(s^2+1)}\left[(s-j)e^{-s\pi}e^{-j\pi} - (s+j)e^{-s\pi}e^{j\pi} - (s-j)e^{-0} + (s+j)e^{0}\right]$$

$$= \frac{A}{2j(s^2+1)}\left[-(s-j)e^{-s\pi} + (s+j)e^{-s\pi} - (s-j) + (s+j)\right] = \frac{A}{2j(s^2+1)}\left(2je^{-\pi s} + 2j\right)$$

$$\Rightarrow A/(s^2+1)(e^{-\pi s}+1)$$

(16) Determine laplace transform of



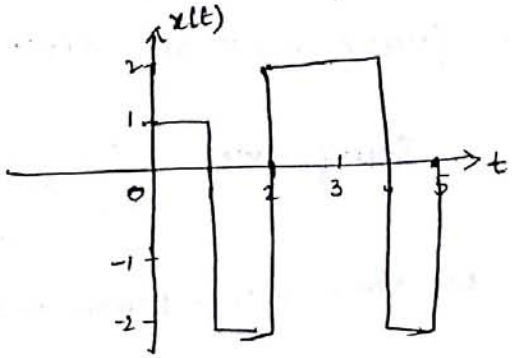
Sol

$$x(t) = \frac{2At}{T} \; ; \; 0<t<T/2$$

$$= 2A - \frac{2At}{T} \quad T/2<t<T.$$

$$L\{x(t)\} = X(s) = \frac{2A}{Ts^2}\left(1 - e^{-sT/2}\right)^2.$$

(17) Determine laplace transform of



Sol

$$x(t) = \begin{cases} 1 & \text{for } 0<t<1 \\ -2 & \text{for } 1<t<2 \\ 2 & \text{for } 2<t<4 \\ -2 & \text{for } 4<t<5 \\ 0 & \text{for } t>5 \end{cases}$$

$$L\{x(t)\} = X(s) = \int_0^1 e^{-st}\,dt + \int_1^2 -2e^{-st}\,dt + \int_2^4 2e^{-st}\,dt + \int_4^5 -2e^{-st}\,dt$$

$$= \left.\frac{e^{-st}}{-s}\right]_0^1 - 2\left[\frac{e^{-st}}{-s}\right]_1^2 + 2\left[\frac{e^{-st}}{-s}\right]_2^4 - 2\left[\frac{e^{-st}}{-s}\right]_4^5$$

$$= \frac{e^{-s}}{-s} + \frac{1}{s} + \frac{2e^{-2s}}{s} - \frac{e^{-s}}{s} - \frac{2e^{-4s}}{s} + \frac{2e^{-2s}}{s} + \frac{2e^{-5s}}{s} - \frac{2e^{-4s}}{s}$$

$$= \frac{1}{s}\left[1 - 2e^{-s} + 4e^{-2s} - 4e^{-4s} + 2e^{-5s}\right]$$

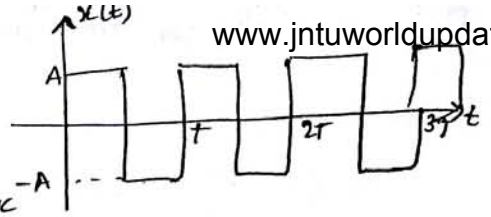
(18) Determine the Laplace transform of $\delta(t)$.

Sol

$$\delta(t) = 1 \text{ for } t=0$$
$$= 0 \text{ otherwise.}$$

$$L\{x(t)\} = X(s) = \left.\frac{e^{-st}}{-s}\right|_{t=0} =$$

Determine the Laplace transform of periodic square wave



Sol) The given waveform satisfy the condition

$x(t+nT) = x(t)$ and so it is periodic

$x_1(t) = A$ for $t = 0$ to $T/2$

$= -A$ for $t = T/2$ to T

From periodicity property of Laplace transform

If $X(S) = L\{x(t)\}$ and if $x(t) = x(t+nT)$ then $X(S) = \frac{1}{1-e^{-ST}} \int_0^T x_1(t) e^{-st} dt$

$$\therefore L\{x(t)\} = X(S) = \frac{1}{1-e^{-ST}} \int_0^T x_1(t) e^{-st} dt$$

we know $x_1(t) =$ Laplace transform $= \frac{A}{S}\left[1 - e^{-ST/2}\right]^2$

$$X(S) = \frac{1}{1-e^{-ST}} \left[\frac{A}{S}\left(1-e^{-ST/2}\right)\right]$$

$$= \frac{1}{\left(1-e^{-ST/2}\right)\left(1+e^{-ST/2}\right)} \left[\frac{A}{S}\left(1-e^{-ST/2}\right)^2\right]$$

$$= \frac{A}{S}\left[\frac{1-e^{-ST/2}}{1+e^{-ST/2}}\right]$$

Initial Value Theorem:

The initial value theorem states that, if $x(t)$ and its derivative are laplace transformable then $\underset{t\to 0}{Lt}\, x(t) = \underset{s\to\infty}{Lt}\, sX(s)$.

i.e $x(0) = \underset{t\to 0}{Lt}\, x(t) = \underset{s\to\infty}{Lt}\, sX(s)$.

Proof

We know that

$$L\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0)$$

On taking limits $s\to\infty$ on both sides of equation.

$$\underset{s\to\infty}{Lt}\, L\left\{\frac{dx(t)}{dt}\right\} = \underset{s\to\infty}{Lt}\left[sX(s) - x(0)\right]$$

$$\Rightarrow \underset{s\to\infty}{Lt} \int_0^\infty \frac{dx(t)}{dt} e^{-st}\, dt = \underset{s\to\infty}{Lt}\left[sX(s) - x(0)\right]$$

$$\Rightarrow \int_0^\infty \frac{dx(t)}{dt}\left(\underset{s\to\infty}{Lt}\, e^{-st}\right) dt = \underset{s\to\infty}{Lt}\, sX(s) - x(0)$$

$$\Rightarrow 0 = \underset{s\to\infty}{Lt}\, sX(s) - x(0)$$

$$\therefore x(0) = \underset{s\to\infty}{Lt}\, sX(s).$$

$$\boxed{\therefore \underset{t\to 0}{Lt}\, x(t) = \underset{s\to\infty}{Lt}\, sX(s)}$$

Final value theorem:

The final value theorem states that if $x(t)$ and its deriative are laplace transformable then $\underset{t\to\infty}{Lt}\, x(t) = \underset{s\to 0}{Lt}\, sX(s)$.

i.e Final value of signal $x(\infty) = \underset{t\to\infty}{Lt}\, x(t) = \underset{s\to 0}{Lt}\, sX(s)$.

Proof

$$L\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0)$$

By taking limits $s\to 0$ on both sides of above eqn we get

$$\underset{s\to 0}{Lt}\, L\left\{\frac{dx(t)}{dt}\right\} = \underset{s\to 0}{Lt}\left\{sX(s) - x(0)\right\}$$

$$\lim_{s\to 0} \int_0^\infty \frac{dx(t)}{dt} e^{-st} dt = \lim_{s\to 0} \left[sx(s) - x(0) \right]$$

$$\int_0^\infty \frac{dx(t)}{dt} \left(\lim_{s\to 0} e^{-st} \right) dt = \lim_{s\to 0} sx(s) - x(0)$$

Not fn of 's'

Z - TRANSFORMS :

FUNDAMENTAL DIFFERENCES BETWEEN CONTINUOUS & DISCRETE TIME SIGNAL
(Ref Unit - I page No: 9)

→ DISCRETE TIME SIGNAL REPRESENTATION USING COMPLEX EXPOENTIAL AND SINUSOIDAL COMPONENTS.
(Ref Unit - I - page No: 10, 11, 12)

→ PERIODICITY PROPERTIES OF DT COMPLEX FUNCTIONS:

→ A DT complex exponential is periodic if it repeats after certain 'N' number of samples.

consider $x(n) = e^{j\omega_0 n}$

$$\therefore x(n+N) = e^{j\omega_0 (n+N)}$$
$$= e^{j\omega_0 n} \cdot e^{j\omega_0 N}$$

For periodicity $x(n) = x(n+N)$

i.e $\cancel{e^{j\omega_0 n}} = \cancel{e^{j\omega_0 n}} \cdot e^{j\omega_0 N} \Rightarrow e^{j\omega_0 N} = 1$.

Expressing $e^{j\omega_0 N}$ in terms of sinusoidal functions

$e^{j\omega_0 N} = \cos\omega_0 N + j\sin\omega_0 N$ by Euler's identity $e^{j\theta} = \cos\theta + j\sin\theta$

→ For periodicity $e^{j\omega_0 N} = 1$ gives $\cos\omega_0 N + j\sin\omega_0 N = 1$. This eqn is satisfied for $\omega_0 N = 2\pi, 4\pi, 6\pi, 8\pi \ldots$

$= 2\pi K$ where K = integer.

(or) $\frac{\omega_0}{2\pi} = \frac{K}{N}$. → (1)

but $\omega_0 = 2\pi f \Rightarrow \frac{\omega_0}{2\pi} = f$ → (2)

Substitute (2) in (1)

$f = \frac{K}{N}$.

$\therefore$ $\boxed{\text{For complex exponential to be periodic, } \frac{\omega_0}{2\pi} = \frac{K}{N} \text{ (i.e rational)}}$

Z-TRANFORM (CONCEPT):

The z-transform of $x(n)$ is denoted by $X(z)$ and defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$ → where z is complex variable.

→ $x(n)$ & $X(z)$ is called z-transform pair represented as

$$x(n) \xleftrightarrow{z} X(z).$$

→ Purpose of learning z-transform is to analyse the DT signals and systems, digital filter design and for synthesis of digital filter / systems.

→ For any input sequence, the z-transform is complex. It has real & imaginary parts.

$$x(n) \longrightarrow \boxed{\text{z-transform}} \longrightarrow X(z)$$

Real & Imaginary parts.

DISTINCTION BETWEEN LAPLACE, FOURIER AND Z TRANSFORM:

z-transform given as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \text{ROC}: r_2 < |z| < r_1$$

where 'z' is defined as $z = re^{j\Omega}$ in which 'r' is magnitude of z i.e $|z|$ and Ω = angle of z i.e $\angle z$

→ putting for $z = re^{j\Omega}$ in $X(z)$ we get

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) (re^{j\Omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [x(n) r^{-n}] e^{-j\Omega n} \rightarrow (1)$$

→ Fourier transform is given by $X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} \rightarrow (2)$ comparing $X(\Omega)$ with $X(z)$ we find that $X(z)$ indicates fourier transform of $x(n) r^{-n}$.

→ Let $X(z)$ of eq ① is evaluated on unit circle. Then $|z| = r = 1$ i.e $r^{-n} = 1$

$$\therefore \quad X(z)\big|_{z=e^{j\Omega}} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}.$$

→ Fourier transform by definition

$$\boxed{X(z)\big|_{z=e^{j\Omega}} = X(\Omega)}$$ → R/Betn FT & Z-T

→ Laplace Transform of i/p sgl $x(t)$ defined as

$$L\{x(t)\} = X(S) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt \rightarrow ①$$

On substituting $S = \sigma + j\Omega$ in above eqn ①

$$L\{x(t)\} = X(S) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\Omega)t}\, dt \rightarrow ②$$

but definition of fourier transform of $x(t)$ is

$$F\{x(t)\} = X(j\Omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\Omega t}\, dt \rightarrow ③$$

By comparing equations ② & ③, we can say that

By putting $\sigma = 0$ in laplace transform we get fourier transform of continuous time signal.

$$\therefore \quad X(j\Omega) = X(S)\big|_{S=j\Omega}.$$

REGION OF CONVERGENCE IN Z-TRANSFORM:

→ Since z-transform is an infinite power series, it exists only for those values of z for which the series converges.

→ The ROC of $X(z)$ is the set of all values of z, for which $X(z)$ attains a finite value.

CONSTRAINTS ON ROC FOR VARIOUS CLASSES OF SIGNALS: (OR)

PROPERTIES OF ROC with proofs:

(1) The ROC is a ring or disk in the z-plane centered at the origin.

(2) The ROC cannot contain any poles.

(3) If $x(n)$ is a finite duration, causal sequence then the ROC is the entire z-plane except at $z=0$.

(4) If $x(n)$ is a finite duration, anti-causal sequence then the ROC is the entire z-plane except at $z=\infty$

(5) If $x(n)$ is a finite duration, two sided sequence then the ROC is entire z-plane except at $z=0$ & $z=\infty$.

(6) If $x(n)$ is an infinite duration, two sided sequence the ROC will consist of a ring in z-plane, bounded on interior and exterior by a pole not containing any poles.

(7) The ROC of an LTI stable system contains unit circle.

(8) The ROC must be connected region.

Case i) Finite duration, right sided (causal) signal:

Let $x(n)$ be finite duration signal with N-samples, defined in range $0 \le n \le (N-1)$

$$\therefore x(n) = \{x(0), x(1), x(2) \ldots x(N-1)\}$$

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

$$= x(0) + x(1) z^{-1} + \ldots + x(N-1) z^{-(N-1)}$$

$$= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \ldots + \frac{x(N-1)}{z^{N-1}}$$

When $z=0$, all the terms except the first term become infinite.

Hence the $X(z)$ exists for all values of z except $z=0$.

$\therefore$ ROC for finite duration right sided is entire z-plane except $z=0$

Case ii) finite duration, left sided (anticausal) signal:

Let $x(n)$ be finite duration signal with N-samples, defined in the range $-(N-1) \le n < 0$

$$\therefore x(n) = \{x(-N-1), \ldots x(-2), x(-1), x(0)\}$$

z-transform of $x(n)$ is

$$X(z) = \sum_{n=-(N-1)}^{0} x(n) z^{-n}$$

$$X(z) = x(-(N-1)) z^{(N-1)} + \ldots + (x(-2)) z^{2} + x(-1) z + x(0)$$

when $z = \infty$, all terms except the last term become infinite. Hence the $X(z)$ exists for all values of z, except $z = \infty$.

$\therefore$ ROC of $X(z)$ is entire z-plane except $z = \infty$.

Case iii) Finite duration, two sided (non-causal) signal:

Let $x(n)$ be a finite duration signal with N-samples, defined in the range $-M \le n \le M$ where $M = \frac{N-1}{2}$.

$$\therefore x(n) = \{x(-M), \ldots, x(-2), x(-1), x(0), x(1), x(2) \ldots x(M)\}$$

z-transform of $x(n)$ is

$$X(z) = \sum_{n=-M}^{M} x(n) z^{-n}$$

$$= x(-M) z^{M} + \ldots + x(-2) z^{2} + x(-1) z + x(0) + x(1) z^{-1} + x(2) z^{-2} + \ldots + x(M) z^{-M}$$

$$= x(-M) z^{M} + \ldots + x(-2) z^{2} + x(-1) z + x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^{2}} + \ldots + \frac{x(M)}{z^{M}}.$$

when $z = 0$, the terms with negative power of z attain infinity and when $z = \infty$, the terms with positive power of z attain infinity. Hence $X(z)$ converges for all values of z, except at $z = 0$ & $z = \infty$.

ROC is entire zplane except $z = 0$ & $z = \infty$.

Case iv) Infinite duration, right sided (causal) signal :

Let $x(n) = r_1^n \; ; n \geq 0$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} r_1^n z^{-n} = \sum_{n=0}^{\infty} (r_1 z^{-1})^n$$

If $0 < |r_1 z^{-1}| < 1$, then $\sum_{n=0}^{\infty} (r_1 z^{-1})^n = \frac{1}{1 - r_1 z^{-1}}$

{ Using infinite geometric series sum formula

$$\sum_{n=0}^{\infty} c_n = \frac{1}{1-c}$$

if $0 < |c| < 1$

$$\therefore X(z) = \frac{1}{1 - \frac{r_1}{z}} = \frac{z}{z - r_1}$$

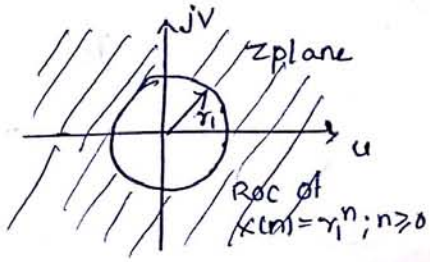
Condition to be satisfied for convergence of $X(z)$ is

$0 < |r_1 z^{-1}| < 1$

$\therefore |r_1 z^{-1}| < 1$

$\frac{|r_1|}{|z|} < 1 \Rightarrow |z| > |r_1|$

represent a circle of radius r_1 in z-plane.



$\rightarrow$ $X(z)$ converges for all points external to the circle of radius r_1 in z-plane.

$\therefore$ ROC of $X(z)$ is exterior of the o^{le} of radius r_1 in z-plane.

Case v) : Infinite duration, left sided (anticausal) signal :

Let $x(n) = r_2^n \; ; n \leq 0$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{0} r_2^n z^{-n} = \sum_{n=0}^{\infty} (r_2^{-1} z)^n$$

If $0 < |r_2^{-1} z| < 1$ then $\sum_{n=0}^{\infty} (r_2^{-1} z)^n = \frac{1}{1 - r_2^{-1} z}$

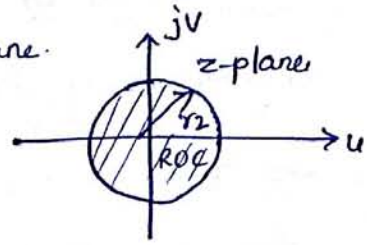
$$\therefore X(z) = \frac{1}{1 - r_2^{-1} z} = \frac{1}{1 - \frac{z}{r_2}} = \frac{r_2}{r_2 - z}$$

$$= \frac{-r_2}{z - r_2}$$

Condition to be satisfied for convergence of $X(z)$ is

$0 < |r_2^{-1} z^{-1}| < 1 \Rightarrow \therefore |r_2^{-1} z| < 1 \Rightarrow \frac{|z|}{|r_2|} < 1 \Rightarrow |z| < |r_2|$

The term $|r_2|$ represent a ckt of radius of r_2 in z-plane. $X(z)$ converges for all pts internal to the ckt of radius r_2 in z-plane. $\therefore$ ROC of $X(z)$ is interior of the circle of radius r_2.



Case vi) Infinite duration, two sided (anticausal) signal:

Let $x(n) = r_1^n u(n) + r_2^n u(-n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} r_2^n z^{-n} + \sum_{n=0}^{\infty} r_1^n z^{-n}$$

$$= \sum_{n=0}^{\infty} r_2^{-n} z^n + \sum_{n=0}^{\infty} r_1^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (r_2^{-1} z)^n + \sum_{n=0}^{\infty} (r_1 z^{-1})^n$$

$$= \frac{1}{1-r_2^{-1}z} + \frac{1}{1-r_1 z^{-1}}$$

{ Using infinite geometric series sum formula if $0<|r_2^{-1}z|<1$ & $0<|r_1 z^{-1}|<1$

The term $\sum_{n=0}^{\infty} (r_2^{-1} z)^n$ converges if

$$0<|r_2^{-1}z|<1$$

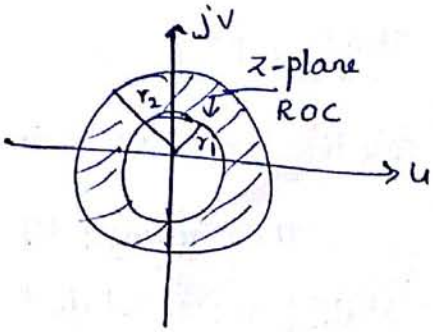
$$\therefore |r_2^{-1}z|<1 \Rightarrow \underline{|r_2^{-1}z|} \Rightarrow \frac{|z|}{|r_2|}<1 \Rightarrow |z|<|r_2|.$$

The term $\sum_{n=0}^{\infty} (r_1 z^{-1})^n$ converges if

$$0<|r_1^{-1}z|<1$$

$$\therefore |r_1 z^{-1}|<1$$

$$\frac{|r_1|}{|z|}<1 \Rightarrow |z|>|r_1|.$$



$\therefore$ ROC is the region between two circles of radius r_1 & r_2.

PROPERTIES OF Z-TRANSFORM:

Linearity Property:

→ States that z-transform of linear weighted combination of discrete time signals is equal to similar linear weighted combination of z transform of individual discrete time signals.

Let $Z[x_1(n)] = X_1(z)$ and $Z[x_2(n)] = X_2(z)$ then by linearity property

$$Z[a_1 x_1(n) + a_2 x_2(n)] = a_1 X_1(z) + a_2 X_2(z) \quad (a_1, a_2 \text{ are constants}).$$

proof:

z-transform definition

$$X_1(z) = Z\{x_1(n)\} = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n}$$

$$X_2(z) = Z\{x_2(n)\} = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$Z[a_1 x_1(n) + a_2 x_2(n)] = \sum_{n=-\infty}^{\infty} [a_1 x_1(n) + a_2 x_2(n)] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [a_1 x_1(n) z^{-n} + a_2 x_2(n) z^{-n}]$$

$$= \sum_{n=-\infty}^{\infty} a_1 x_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} a_2 x_2(n) z^{-n}$$

$$= a_1 \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$= a_1 X_1(z) + a_2 X_2(z).$$

Shifting property:

Case i) Two sided z-transform

The shifting property of z-transform states that z-transform of a shifted signal shifted by 'm' units of time is obtained by multiplying z^m to z-transform of unshifted signal.

Let $Z\{x(n)\} = X(z)$ then $Z\{x(n-m)\} = z^{-m} X(z)$ & $Z\{x(n+m)\} = z^{m} X(z)$

Proof :

$$X(z) = z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$z\{x(n-m)\} = \sum_{n=-\infty}^{\infty} x(n-m) z^{-n}$$

Let $n-m=p \therefore n=p+m$

when $n \to -\infty, p \to -\infty$
$n \to \infty, \; p \to \infty$

$$= \sum_{p=-\infty}^{\infty} x(p) z^{-(m+p)}$$

$$= \sum_{p=-\infty}^{\infty} x(p) z^{-p} \cdot z^{-m}$$

$$= z^{-m} \sum_{p=-\infty}^{\infty} x(p) z^{-p} = z^{-m} X(z)$$

$$z\{x(n+m)\} = \sum_{n=-\infty}^{\infty} x(n+m) z^{-n}$$

$$= \sum_{p=-\infty}^{\infty} x(p) z^{-(p-m)}$$

$$= \sum_{p=-\infty}^{\infty} x(p) z^{-p} \cdot z^{m}$$

$$= z^{m} \sum_{p=-\infty}^{\infty} x(p) z^{-p} \qquad (\because \text{ if } p \to n)$$

$$= z^{m} \cdot X(z).$$

Case ii) One sided z-transform :

Let $x(n)$ be discrete time signal defined in range $0 < n < \infty$

$$z[x(n)] = X(z)$$

By shifting property

$$z[x(n-m)] = z^{-m} X(z) + \sum_{i=1}^{m} x(-i) z^{-(m-i)}$$

$$z[x(n+m)] = z^{m} X(z) - \sum_{i=0}^{m-1} x(i) z^{(m-i)}$$

Proof

$$X(z) = z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$z\{x(n-m)\} = \sum_{n=0}^{\infty} x(n-m) z^{-n}$$

multiply by z^m & z^{-m}

$$= \sum_{n=0}^{\infty} x(n-m) z^{-n} z^{m} \cdot z^{-m}$$

$$= z^{-m} \sum_{n=0}^{\infty} x(n-m) z^{-(n-m)}$$

$$= z^{-m} \sum_{p=-m}^{\infty} x(p) z^{-p} \quad \{ \text{Let } n-m=p$$

$$= z^{-m} \sum_{p=0}^{\infty} x(p) z^{-p} + \sum_{p=-m}^{-1} x(p) z^{-p} \cdot z^{-m}$$

$$= z^{-m} \sum_{p=0}^{\infty} x(p) z^{-p} + z^{-m} \sum_{p=1}^{m} x(-p) z^{p}$$

Let $p=n$, in 1st summation
$p=i$ in 2nd "

$$= z^{-m} \sum_{n=0}^{\infty} x(n) z^{-n} + z^{-m} \sum_{i=1}^{m} x(-i) z^{i}$$

$$= z^{m} X(z) + \sum_{i=1}^{m} x(-i) z^{-(m-i)}$$

$$z\{x(n+m)\} = \sum_{n=0}^{\infty} x(n+m) z^{-n} = \sum_{n=0}^{\infty} x(n+m) z^{-n} z^{m} z^{-m} \quad (\text{multiply by } z^m \text{ \& } z^{-m})$$

$$= z^{m} \sum_{n=0}^{\infty} x(n+m) z^{-(n+m)}$$

$$= z^{m} \sum_{p=m}^{\infty} x(p) z^{-p}$$

Let $n+m=p$
when $n \to 0$, $p \to m$
$n \to \infty$, $p \to \infty$

$$= z^{m} \sum_{p=0}^{\infty} x(p) z^{-p} - z^{m} \sum_{p=0}^{m-1} x(p) z^{-p}$$

$$= z^{m} \sum_{n=0}^{\infty} x(n) z^{-n} - z^{m} \sum_{i=0}^{m-1} x(i) z^{-i}$$

$$= z^{m} X(z) - \sum_{i=0}^{m-1} x(i) z^{m-i} \quad \{ \because \text{Let } p=n \text{ in } 1^{st},\ p=i \text{ in } 2^{nd}.$$

(3) Multiplication by 'n' (or Differentiation in z-domain)

Proof

If $Z[x(n)] = X(n)$ then $Z[n\,x(n)] = -z\frac{d}{dz}X(z)$

In general $Z[n^m x(n)] = \left(-z\frac{d}{dz}\right)^m X(z)$

$$X(z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z[n\,x(n)] = \sum_{n=-\infty}^{\infty} n\,x(n) z^{-n} = \sum_{n=-\infty}^{\infty} n\,x(n) z^{-n} z\, z^{-1}$$

$$= -z \sum_{n=-\infty}^{\infty} x(n)\left[-n z^{-n-1}\right]$$

$$= -z \sum_{n=-\infty}^{\infty} x(n)\left[\frac{d}{dz} z^{-n}\right] \qquad \left\{ \because \frac{d}{dz} z^{-n} = -n z^{-n-1} \right.$$

$$= -z \frac{d}{dz} \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z[n\,x(n)] = -z\frac{d}{dz}X(z). \qquad \left\{ \text{Interchanging summation \& differentiation} \right.$$

(4) Multiplication by an exponential sequence, a^n (Scaling in z-domain)

If $Z\{x(n)\} = X(z)$ then $Z\{a^n x(n)\} = X(a^{-1}z)$

Proof

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z[a^n x(n)] = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (a^{-1} z)^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (a^{-1} z)^{-n}$$

$$= X(a^{-1} z).$$

(5) Time Reversal :

If $z\{x(n)\} = X(z)$ then $z\{x(-n)\} = X(z^{-1})$

proof

$$z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

$$= \sum_{p=-\infty}^{\infty} x(p) z^{p}$$

$$= \sum_{p=-\infty}^{\infty} x(p) (z^{-1})^{-p}$$

$$= X(z^{-1})$$

Let $p = -n$
when $n \to -\infty$, $p \to \infty$
$n \to \infty$, $p \to -\infty$

(6) Conjugation :

If $z\{x(n)\} = X(z)$ then $z\{x^*(n)\} = X^*(z^*)$

proof

$$X(z) = z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$z\{x^*(n)\} = \sum_{n=-\infty}^{\infty} x^*(n) z^{-n}$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n) (z^*)^{-n}\right]^*$$

$$= [X(z^*)]^*$$

$$= X^*(z^*)$$

INVERSE Z-TRANSFORM:

Let $X(z)$ be z-transform of discrete time signal $x(n)$. The inverse z-transform is the process of recovering the discrete time sgl $x(n)$ from its z-transform $X(z)$.

→ The signal $x(n)$ can be uniquely determined from $X(z)$ & its ROC.

It can be determined by three methods

(i) Direct evaluation by contour integration (or residue method)

(ii) partial fraction expansion method

(iii) power series expansion method.

① Determine the inverse z-transform of fn $X(z) = \dfrac{3+2z^{-1}+z^{-2}}{1-3z^{-1}+2z^{-2}}$ by following three methods and prove that it is unique.

Sol (i) Residue Method:

$$X(z) = \frac{3+2z^{-1}+z^{-2}}{1-3z^{-1}+2z^{-2}} = \frac{\cancel{z^{-2}}(3z^2+2z+1)}{\cancel{z^{-2}}(z^2-3z+2)}$$

$$\therefore X(z) = \frac{3z^2+2z+1}{z^2-3z+2}$$

$$\begin{array}{r|l} & 3 \\ \hline z^2-3z+2 & \cancel{3z^2}+2z+1 \\ & \underline{\pm\cancel{3z^2} \mp 9z \pm 6} \\ & 11z-5 \end{array}$$

$$= 3 + \frac{11z-5}{z^2-3z+2}$$

$$= 3 + \frac{11z-5}{(z-1)(z-2)}$$

Let $X_1(z) = 3$, $X_2(z) = \dfrac{11z-5}{(z-1)(z-2)}$; i.e $X(z) = X_1(z) + X_2(z)$

$$x(n) = z^{-1}\{X(z)\} = z^{-1}\{X_1(z)\} + z^{-1}\{X_2(z)\}$$

$$= z^{-1}\{3\} + z^{-1}\left\{\frac{11z-5}{(z-1)(z-2)}\right\}$$

$$= 3\delta(n) + \sum_{i=1}^{N}\left[(z-p_i)X_2(z)z^{n-1}\right]_{z=p_i}$$ Using residue theorem.

$$= 3\delta(n) + (z-1)\left.\frac{11z-5 \cdot z^{n-1}}{(z-1)(z-2)}\right|_{z=1} + (z-2)\cdot\left.\frac{11z-5}{(z-1)(z-2)}\cdot z^{n-1}\right|_{z=2}$$

$$= 3\delta(n) + \frac{11-5}{1-2}(1)^{n-1} + \frac{11\times 2-5}{2-1}\,2^{n-1}$$

$$\therefore\ x(n) = 3\delta(n) - 6u(n-1) + 17(2)^{n-1}u(n-1)$$

$$= 3\delta(n) + \left[-6 + 17(2)^{n-1}\right]u(n-1).$$

when $n=0$, $x(0) = 3-0+0 = 3$

when $n=1$, $x(1) = 0-6+17\times 2^0 = 11$

when $n=2$, $x(2) = 0-6+17\times 2^1 = 28$

when $n=3$ $x(3) = 0-6+17\times 2^2 = 62$

$$\therefore\ x(n) = \{3, 11, 28, 62, 130 \cdots\}$$

Method 2:

$$x(z) = \frac{3+2z^{-1}+z^{-2}}{1-3z^{-1}+2z^{-2}} = \frac{3z^2+2z+1}{z^2-3z+2}$$

$$\therefore\ \frac{x(z)}{z} = \frac{3z^2+2z+1}{z(z-1)(z-2)}$$

Let $\frac{x(z)}{z} = \frac{3z^2+2z+1}{z(z-1)(z-2)} = \frac{A_1}{z} + \frac{A_2}{z-1} + \frac{A_3}{z-2}$

$$A_1 = z\cdot\left.\frac{x(z)}{z}\right|_{z=0} = \cancel{z}\cdot\left.\frac{3z^2+2z+1}{\cancel{z}(z-1)(z-2)}\right|_{z=0} = 0.5$$

$$A_2 = (z-1)\left.\frac{x(z)}{z}\right|_{z=1} = (z-\cancel{1})\left.\frac{3z^2+2z+1}{z(\cancel{z-1})(z-2)}\right|_{z=1} = -6$$

$$A_3 = (z-2)\left.\frac{x(z)}{z}\right|_{z=2} = (z-\cancel{2})\cdot\left.\frac{3z^2+2z+1}{z(z-1)(z-\cancel{2})}\right|_{z=2} = 8.5$$

$$\frac{x(z)}{z} = \frac{0.5}{z} - \frac{6}{z-1} + \frac{8.5}{z-2}.$$

$$\therefore X(z) = 0.5 - 6 \cdot \frac{z}{z-1} + 8.5 \frac{z}{z-2}$$

On taking inverse z-transform

$$x(n) = 0.5\,\delta(n) - 6 \cdot u(n) + 8.5\,(2)^n u(n)$$

$$= 0.5\,\delta(n) + \left[-6 + 8.5(2)^n\right] u(n)$$

When $n=0$, $x(0) = 0.5 - 6 + 8.5 \times 2^0 = 3$

$n=1$, $x(1) = 0 - 6 + 8.5 \times 2^1 = 11$

$n=2$ $x(2) = 0 - 6 + 8.5 \times 2^2 = 28$

$n=3$, $x(3) = 0 - 6 + 8.5 \times 2^3 = 62$

$$\therefore x(n) = \{\underset{\uparrow}{3}, 11, 28, 62 \cdots\}$$

Method 3: Power Series Expansion method.

$$
\begin{array}{r|l}
 & 3 + 11z^{-1} + 28z^{-2} + 62z^{-3} + 130z^{-4} + \cdots \\
\hline
1 - 3z^{-1} + 2z^{-2} & \cancel{3} + 2z^{-1} + z^{-2} \\
 & \cancel{3} - 9z^{-1} + 6z^{-2} \\
 & \quad 11z^{-1} - 5z^{-2} \\
 & \quad \cancel{11z^{-1}} - 33z^{-2} + 22z^{-3} \\
 & \qquad 28z^{-2} - 22z^{-3} \\
 & \qquad \cancel{28z^{-2}} - 84z^{-3} + 56z^{-4} \\
 & \qquad\quad 62z^{-3} - 56z^{-4} \\
 & \qquad\quad \cancel{62z^{-3}} - 186z^{-4} + 124z^{-5} \\
 & \qquad\qquad 130z^{-4} + 124z^{-5} \\
 & \qquad\qquad\quad \vdots
\end{array}
$$

$$\therefore X(z) = 3 + 11z^{-1} + 28z^{-2} + 62z^{-3} + 130z^{-4} + \cdots \quad \rightarrow (1)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \cdots + x(0) + x(1) z^{-1} + x(2) z^{-2} + \cdots \quad \rightarrow (2)$$

On comparing (1) & (2)

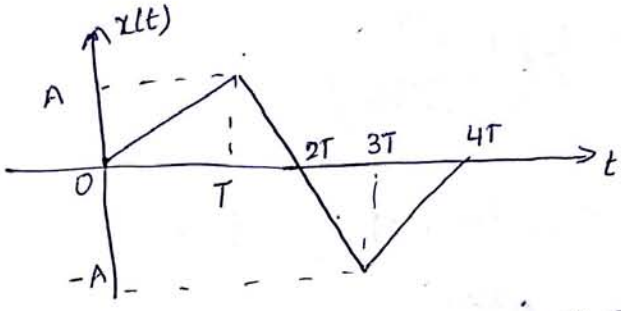
$x(0) = 3$, $x(1) = 11$, $x(2) = 28$, $x(3) = 62$

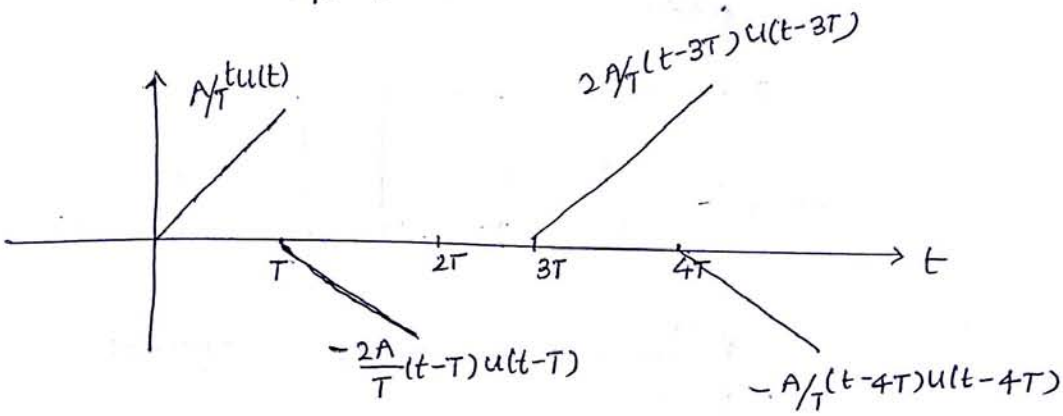
$$\therefore x(n) = \{3, 11, 28, 62 \cdots\}$$

Wave synthesis Using Laplace transform:

To express the function into singular functions and express it in synthesis

① Find the Laplace transform of the waveform.





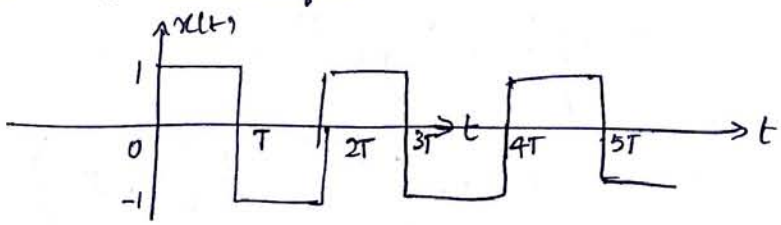
$$x(t) = \frac{A}{T} t\, u(t) - \frac{2A}{T}(t-T)\,u(t-T) + \frac{2A}{T}(t-3T)\,u(t-3T) - \frac{A}{T}(t-4T)\,u(t-4T)$$

Taking Laplace transform on both sides

$$X(s) = \frac{A}{T}\frac{1}{s^2} - \frac{2A}{T}\frac{e^{-Ts}}{s^2} + \frac{2A}{T}\frac{e^{-3Ts}}{s^2} - \frac{A}{T}\frac{e^{-4Ts}}{s^2}$$

$$= \frac{A}{Ts^2}\left[1 - 2e^{-Ts} + 2e^{-3Ts} - e^{-4Ts}\right]$$

② Find the Laplace transform



$$x(t) = u(t) - 2u(t-T) + 2u(t-2T) - 2u(t-3T) + 2u(t-4T) - 2u(t-5T) + \cdots$$

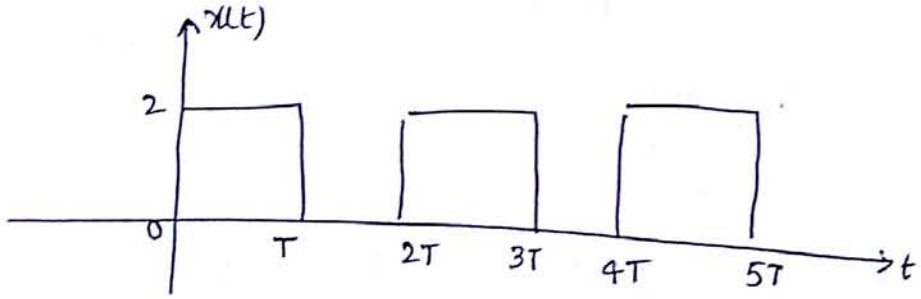
$$X(s) = \frac{1}{s} - \frac{2e^{-Ts}}{s} + \frac{2e^{-2Ts}}{s} - \frac{2e^{-3Ts}}{s} + \frac{2e^{-4Ts}}{s} - \frac{2e^{-5Ts}}{s} + \cdots$$

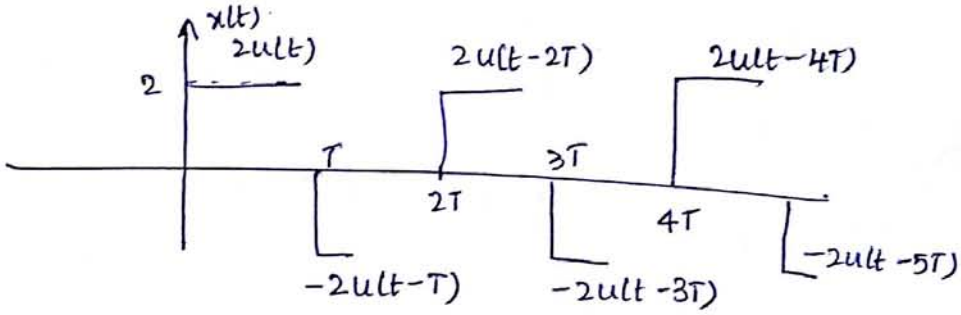
$$= \frac{1}{s}\left[1 - 2e^{-Ts}(1 - e^{-Ts} + e^{-2Ts} - e^{-3Ts} + e^{-4Ts} + \cdots\right]$$

$$= \frac{1}{s}\left[1 - 2e^{-Ts}(1+e^{-Ts})^{-1}\right] = \frac{1}{s}\left(1 - \frac{2e^{-Ts}}{1+e^{-Ts}}\right) = \frac{1}{s}\left(\frac{1+e^{-Ts} - 2e^{-Ts}}{1+e^{-Ts}}\right)$$

$$= \frac{1}{s}\left(\frac{1-e^{-Ts}}{1+e^{-Ts}}\right)$$

(3) Find the Laplace transform of waveform





$$x(t) = 2u(t) - 2u(t-T) + 2u(t-2T) - 2u(t-3T) + 2u(t-4T) - 2u(t-5T) \cdots$$

Taking Laplace transform

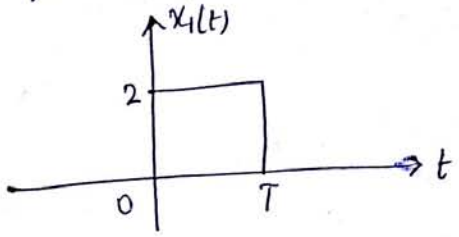
$$X(s) = \frac{2}{s} - \frac{2e^{-Ts}}{s} + \frac{2e^{-2Ts}}{s} - \frac{2e^{-3Ts}}{s} + \frac{2e^{-4Ts}}{s} - \frac{2e^{-5Ts}}{s} \cdots$$

$$= \frac{2}{s}\left[1 - e^{-Ts} + e^{-2Ts} - e^{-3Ts} + e^{-4Ts} - e^{-5Ts} \cdots\right]$$

$$= \frac{2}{s}\left[1 + e^{-Ts}\right]^{-1} = \frac{2}{s}\left(\frac{1}{1+e^{-Ts}}\right)$$

Another method

The given waveform is periodic with period 2T.



periodicity property

$$X(s) = \frac{1}{1-e^{-2Ts}} X_1(s)$$

$$x_1(t) = 2\left[u(t) - u(t-T)\right]$$

$$X_1(s) = 2\left(\frac{1}{s} - \frac{e^{-Ts}}{s}\right) = \frac{2}{s}\left(1-e^{-Ts}\right)$$

$$X(s) = \frac{1}{1-e^{-2Ts}}\left(\frac{2}{s}\right)\left(1-e^{-Ts}\right)$$

$$= \frac{2}{s} \cdot \frac{1-e^{-Ts}}{(1+e^{-Ts})(1-e^{-Ts})} = \frac{2}{s}\left(\frac{1}{1+e^{-Ts}}\right)$$

Periodicity property:

Used in determining the transform of periodic time functions.

$x(t) = x(t+nT)$. where T is period, $n = 0, 1, 2 \ldots$

$$X(s) = \int_0^\infty x(t) e^{-st}\, dt.$$

$$X(s) = \int_0^T x(t) e^{-st} + \int_T^{2T} x(t) e^{-st} + \int_{2T}^{3T} x(t) e^{-st} dt + \cdots + \int_{nT}^{(n+1)T} x(t) e^{-st} dt + \cdots$$

$$= \int_0^T x(t) e^{-st} dt + e^{-sT} \int_0^T x(t+T) e^{-st} dt + \cdots + e^{-nsT} \int_0^T x(t+nT) e^{-st} dt$$

$$x(t) = x(t+T) = x(t+2T) \ldots$$

$$X(s) = \int_0^T x(t)\, e^{-st} dt + e^{-sT} \int_0^T x(t)\, e^{-st} dt + \cdots + e^{-nsT} \int_0^T x(t)\, e^{-st} dt + \cdots$$

$$= \left[1 + e^{-sT} + e^{-2sT} + \cdots + e^{-nsT}\right] \int_0^T x(t)\, e^{-st} dt$$

$$= \left[1 - e^{-sT}\right]^{-1} \int_0^T x(t)\, e^{-st} dt = \frac{1}{1 - e^{-sT}} X_1(s).$$

where $X_1(s) = \int_0^T x(t)\, e^{-st}\, dt$