DEPARTMENT OF ELECTRONICS AND COMMUNICATION ENGINEERING

Integral Transforms (J224D)

[R20]

B.TECH ECE
(II YEAR – II SEM)
(2022-23)

J. B. INSTITUTE OF ENGINEERING AND TECHNOLOGY
(UGC AUTONOMOUS)
Accredited by NBA & NAAC,
Approved by AICTE & Permanently affiliated to JNTUH
Bhaskar Nagar, Yenkapally(V), Moinabad(M), Ranaga Reddy(D),Hyderabad – 500 075,
Telanagana, India.

AY 2020-21 onwards	**J. B. Institute of Engineering and Technology (UGC Autonomous)**	**B.Tech II Year – II Sem**			
Course Code:	**Integral Transforms (COMMON TO EEE & ECE**	**L**	**T**	**P**	**D**
Credits: 3		**3**	**0**	**0**	**0**

Pre-requisite: Differential Equations

Course Objectives:

This course will enable students to:

1. Approximation of real-valued periodic functions to suitably restricted non-periodic functions f(x) defined for all real numbers
2. How to use Laplace Transform methods to solve ordinary and partial differential equations
3. Make them familiar with the methods of solving differential equations, partial differential equations.
4. The properties of z-transform and associating the knowledge of properties of roc in response to different operations on discrete signals.
5. Discretization techniques to find approximate solutions of differential equations different types of errors involved in such solutions, their measures and practical applications.

Module 1 : Introduction **(8L)**

Basic introduction of the course using precise examples like periodic functions, signal propagation, solving mathematical models corresponding to Electrical Circuits.

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Module 2 : Laplace Transforms **(12L)**

Unit 1: Laplace Transform (LT) – definition – linearity property of LT. Existence Theorem – First and Second Translation theorems. Change of scale property. LT of derivatives – LTs: multiplication by t and division by t – Initial and Final Value theorems.

Unit 2: Inverse Laplace Transforms: definition – standard forms. First and Second shifting theorems. Change of scale property – Use of partial fractions – Multiplication by powers of S, division by S. Inverse transform of derivatives. Heaviside expansion theorem. Inverse Laplace Transform of integrals – definition of convolution – Convolution theorem.

Module 3: Fourier Transforms (10L)

Fourier Transforms – Fourier integral formula, Inverse Theorem for Fourier Transform; Fourier Sine Transform, Inverse formula for Fourier Sine Transform; Fourier Cosine Transform, Inverse formula for Fourier Cosine Transform; linearity property, change of scale property, shifting property, Modulation Theorem.

Module 4: Z-Transforms **(9L)**

Definition and properties of Z-Transform, Standard functions of Z-Transform, Unit step Function. Unit Impulse function, Initial value Theorem and Final value Theorem, Inverse Z-Transform, Partial fraction method, Difference Equation using Z-Transforms.

Module 5: Henkel Transforms **(9L)**

Henkel Transforms- Henkel Transform of the derivatives of a function.- Application of Henkel Transforms in boundary value problems-The finite Henkel Transform

Text Books:

1. "Integral Transforms", A.R.Vashista, Dr. R.K.Gupta, - Krishna Prakasham Mandirurray
2. "Theory and problems of Laplace transforms" .R.Spiegel, Shamus Outline Series Tata Mac Grawhill.
3.

Reference Books:

1. "Integral Transforms & their applications ",Brian Daries, - Springers
2. "Integral Transforms & their Applications", L Debnath , D Bhatta, – Chapman & Hall/CRC
3. "Integral Transforms & their Applications", Chorafas,

E - Resources:

1. **https://nptel.ac.in/content/storage2/courses/112104158/lecture8.pdf**
2. **https://tutorial.math.lamar.edu/classes/de/inversetransforms.aspx**
3. **http://www.thefouriertransform.com/**
4. **http://dsp-book.narod.ru/TAH/ch06.pdf**
5. **https://www.henkel-adhesives.com/in/en.html**

Course Outcomes:

On completion of the course, the students will be able to:

1. Understand the concepts of integral transforms
2. Determine laplace transform of a function and understand the fundamental properties and apply laplace transform in solving odes.
3. Determine fourier and inverse fourier transform of a function and understand the fundamental properties and apply fourier transform in solving odes.
4. *Apply the z transform techniques to solve second-order ordinary difference equations*.
5. Apply the hankel transform in the infinite 2-dimensional plane

CO-PO/PSO Mapping Chart

(3/2/1 indicates strength of correlation)

3 – Strong; 2 – Medium; 1 - Weak

Course Outcomes (COs)	Program Outcomes (POs)												Program Specific Outcomes	
	PO 1	PO 2	PO 3	PO 4	PO 5	PO 6	PO 7	PO 8	PO 9	PO 10	PO 11	PO 12	PSO 1	PSO 2
CO1	3	3	2	3	-	-	-	-	-	-	-	2	-	-
CO2	3	3	2	3	-	-	-	-	-	-	-	2	-	-
CO3	3	3	2	3	-	-	-	-	-	-	-	2	-	-
CO4	3	3	2	3	-	-	-	-	-	-	-	2	-	-
CO5	3	3	2	3	-	-	-	-	-	-	-	2	-	-
Average	3	3	2	3	-	-	-	-	-	-	-	2	-	-

11-09-23

INTEGRAL TRANSFORMS.

MODULE-I : LAPLACE TRANSFORMS.

1) $L\{1\} = \frac{1}{s}$

Der: $L\{1\} = \int_0^\infty e^{-st}\, 1 . dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$ $\quad \left[L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt \right]$

$$= \frac{e^{-\infty} - e^{0}}{-s}$$

$$= \frac{0-1}{-s} = \frac{1}{s}$$

2) $L\{e^{at}\} = \int_0^\infty e^{-st} . e^{at}\, dt$

$$= \int_0^\infty e^{(-s+a)t} . dt$$

$$= \left[\frac{e^{(-s+a)t}}{-s+a}\right]_0^\infty$$

$$= \frac{e^{-\infty} - e^{0}}{-s+a} = \frac{0-1}{-s+a}$$

$$= \frac{1}{s-a}$$

Modelling: Electric circuits.

Modelling Means setting up Mathematical models of physical or other systems. Now we shall model Electric circuits. Their model will be linear Differential equations [LDE]. This discussion will be profitable to students of EEE, ECE, ECM.

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1^{st} step : Modelling.

voltage drop across the resistor $= RI$.

" " " Inductor $= N.\frac{dI}{dt}$

By KVL, the sum of the voltage drops must be equal to the e.m.f, E(t)

$$L.\frac{dI}{dt} + RI = E(t).$$

2^{nd} step : solution of the equation for a special circuit.

$L = 0.1\ H$

$R = 5\Omega$

and 12v battery gives emf.

$$(0.1)\frac{dI}{dt} + 5I = 12.$$

$$\frac{dI}{dt} + 50I = 120 \qquad \left\{\frac{dy}{dx} + Py = Q\right\}$$

$$I.F = e^{\int 50\, dt} = e^{50t}.$$

$$y.(I.F) = \int Q(I.F)\, dx$$

$$I.e^{50t} = \int 120 . e^{50t}\, dt$$

$$= 12\cancel{0} . \frac{e^{50t}}{5\cancel{0}} + c$$

$$= 2.4 + ce^{-50t}.$$

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** gamma

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\rightarrow \Gamma\left(\frac{5}{2}\right) = \left(\frac{5}{2}-1\right) \cdot \Gamma\frac{5-1}{2}$

$= \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}.$

$\rightarrow \Gamma\frac{11}{2} = \left(\frac{11}{2}-1\right)\Gamma\left(\frac{11}{2}-1\right)$

$= \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}.$

$\rightarrow \Gamma\frac{15}{2} = \frac{13}{2} \cdot \frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}.$

$\rightarrow \Gamma 7 = 6!$

$\rightarrow \Gamma 10 = 9!$

FORMULAS :

$L\{1\} = \frac{1}{s}$

$L\{e^{at}\} = \frac{1}{s-a}$

$L\{\sin at\} = \frac{a}{s^2+a^2}$

$L\{\cos at\} = \frac{s}{s^2+a^2}$

$L\{\sinh at\} = \frac{a}{s^2-a^2}$

$L\{\cosh at\} = \frac{s}{s^2-a^2}$

$L\{t^n\} = \frac{n!}{s^{n+1}}$, if n is +ve integer.

$= \frac{\Gamma(n+1)}{s^{n+1}}$, if $n+1>0$.

Ex:

1) $L\{t^8 + 7\sin 6t + 8\cos 5t - e^{7t}\}$

$= \frac{8!}{s^9} + 7 \cdot \frac{6}{s^2+36} + 8 \cdot \frac{s}{s^2+25} - \frac{1}{s-7}.$

2) $L\{t^{5/2} - 6\sinh 3t + 5\cosh 2t - e^{-9t}\}$

$= \frac{\Gamma\frac{7}{2}}{s^{7/2}} - 6\left(\frac{3}{s^2-9}\right) + 5\left(\frac{s}{s^2-4}\right) - \frac{1}{s+9}.$

13/09/23

LAPLACE TRANSFORM:

L·T Method solves Differential Equations and corresponding initial and boundary value problems.

This is an excellent tool for solving Linear Differential Equations.

Def: LT of a f(t) is a linear integral transform which is defined as $\tilde{f}(s) = L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt,\ t \geq 0.$

(Provided the integral exists).

$f(t) \rightarrow$ Time domain function.

$e^{-st} \rightarrow$ kernel of the transform $[K(s,t)]$.

$L \rightarrow$ Laplace transform operator.

$s \rightarrow$ complex variable

$\tilde{f}(s) \rightarrow$ complex Variable function.

- EXISTANCE OF LAPLACE TRANSFORM.

1. f(t) is piecewise continuous.
2. f(t) is of exponential order.

i.e., $\lim_{t\to\infty} e^{-st} f(t) = 0.$

(3m)

Q: Prove that the function $f(t) = t^2$ is of exponential order 3.

$$\lim_{t\to\infty} e^{-3t} \cdot t^2 = \lim_{t\to\infty} \frac{t^2}{e^{3t}} \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{t\to\infty} \frac{2t}{3e^{3t}} \left(\ " \ \right)$$

$$= \lim_{t\to\infty} \frac{2(1)}{9e^{3t}} = \frac{2}{\infty} = 0,$$

$\downarrow$ finite quantity.

Hence, it is ~~(at)~~ exponential order.

Q: Prove that the L.T of e^{t^3} does not exists.

$$\underset{t\to\infty}{Lt}\ e^{-st}\cdot e^{t^3} = \underset{t\to\infty}{Lt}\ e^{t^3-st}$$

$$= e^{\infty}$$

$$= \infty.$$

It is not of exponential order. so LT of e^{t^3} does not exist.

• LINEAR PROPERTY.

$$L\{a\cdot f(t) + b\cdot g(t) + c\cdot h(t)\}$$

$$= a\cdot L\{f(t)\} + b\cdot L\{g(t)\} + c\cdot L\{h(t)\}$$

where a,b,c are constants and f,g,h are functions of t.

Formulas:

1) $L\{(t^2+1)^2\} = L\{t^4+1+2t^2\}$

$$= \frac{4!}{s^5} + \frac{1}{s} + 2\cdot\frac{2!}{s^3}.$$

2) $L\{\sin 2t\cdot\cos t\}$

$$= L\left\{\frac{2\sin 2t\cdot\cos t}{2}\right\} \qquad [2\sin A\cos B = \sin(A+B) + \sin(A-B)]$$

$$= \frac{1}{2}\cdot L\{\sin 3t + \sin t\}$$

$$= \frac{1}{2}\left[\frac{3}{s^2+9} + \frac{1}{s^2+1}\right]$$

3) $L\{\cosh^2 2t\}$

$$= L\left\{\frac{1+\cosh 4t}{2}\right\}$$

$$= \frac{1}{2}\left[L\{1\} + L\{\cosh 4t\}\right] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2-16}\right]$$

4) $L\{\cos^3 2t\}$

$$\cos^3 2t = \frac{1}{4}\cos 6t + \frac{3}{4}\cos 2t$$

$$= \frac{1}{4}\cdot\frac{s}{s^2+36} + \frac{3}{4}\cdot\frac{s}{s^2+4}$$

$$\left[\begin{array}{l}\cos 3a = 4\cos^3 A - 3\cos A \\ 4\cos^3 A = \cos 3A + 3\cos A \\ \cos^3 A = \frac{1}{4}\cos 3A + \frac{3}{4}\cos A\end{array}\right]$$

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14/09/23

- FIRST SHIFTING THEOREM : [FST]

If $L\{f(t)\} = \tilde{f}(s)$, then $L\{e^{at} f(t)\} = \tilde{f}(s-a)$

PROOF : $\tilde{f}(s-a)$

$$= \int_0^{\infty} e^{-(s-a)t} \cdot f(t)\, dt$$

$$= \int_0^{\infty} e^{-st} \cdot [e^{at} \cdot f(t)]\, dt$$

$$= L\{e^{at} \cdot f(t)\}.$$

Q: 1) $L\{e^{-t} \cos 2t\}$

$$L\{\cos 2t\} = \frac{s}{s^2+4} = \tilde{f}(s)$$

Now applying FST, we have

$$L\{e^{-t} \cos 2t\} = \tilde{f}(s+1)$$

$$= \frac{(s+1)}{(s+1)^2+4}$$

2) $L\{e^{-3t}(2\cos 5t - 3\sin 5t)\}$

$L\{2\cos 5t - 3\sin 5t\}$

$$= 2 \cdot \frac{s}{s^2+25} - 3 \cdot \frac{5}{s^2+25} = \tilde{f}(s)$$

Now applying FST, we have

$$L\{e^{-3t}(2\cos 5t - 3\sin 5t)\} = \tilde{f}(s+3)$$

$$= 2 \cdot \frac{(s+3)}{(s+3)^2+25} - 3 \cdot \frac{5}{(s+3)^2+25}$$

$$= 2 \cdot \frac{(s+3)}{(s+3)^2+25} - \frac{15}{(s+3)^2+25}$$

3) $L\{\sqrt{t} \cdot e^{-3t}\}$

$$L\{\sqrt{t}\} = L\{t^{1/2}\}$$

$$= \frac{\Gamma(\frac{1}{2}+1)}{s^{3/2}} = \frac{\Gamma(\frac{3}{2})}{s^{3/2}}$$

$$= \frac{\frac{1}{2} \cdot \Gamma(\frac{1}{2})}{s^{3/2}} = \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} = \tilde{f}(s)$$

Now applying FST, we have

$$L\{\sqrt{t} \cdot e^{-3t}\} = \tilde{f}(s+3)$$

$$= \frac{1}{2} \cdot \frac{\sqrt{\pi}}{(s+3)^{3/2}}$$

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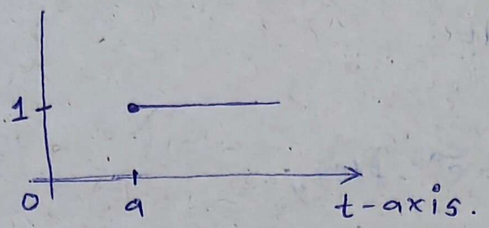
- UNIT STEP FUNCTION : (or)

(HEAVISIDE UNIT FUNCTION).

It is defined as $H(t-a)$ or $u(t-a) = \begin{cases} 0 & \text{if } t<a \\ 1 & \text{if } t>a. \end{cases}$



The graph is a straight line parallel to t-axis from a to ∞.

$$L\{u(t-a)\}$$

$$= \int_0^\infty e^{-st} u(t-a)\,dt.$$

$$= \int_0^a e^{-st}.\, u\cancel{(t-a)}^{0}\,dt + \int_a^\infty e^{-st}.\, \underbrace{u(t-a)}_{1}\,dt$$

$$= 0 + \int_a^\infty e^{-st}.\,1\,dt.$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= \frac{e^{-\infty} - e^{-as}}{-s} = \frac{0 - e^{-as}}{-s}$$

$$= \frac{e^{-as}}{s} \,//$$

- SECOND SHIFTING THEOREM [SST]

If $L\{f(t)\} = \tilde{f}(s)$ and $g(t) = \begin{cases} f(t-a) & \text{if } t>a \\ 0 & \text{if } t<a \end{cases}$, then.

$$L\{g(t)\} = e^{-as}\,\tilde{f}(s).$$

<u>PROOF</u> : By def,

$$L\{g(t)\} = \int_0^\infty e^{-st}.\, g(t)\,dt$$

$$= 0 + \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= \int_0^\infty e^{-st}.\, f(t-a)\,dt$$

Put $t-a=u$

$t = a+u$

$dt = 0+du$

$dt = du$

<u>L.L</u> : $t=a$. $u=t-a$ $u=a-a=0$ //

<u>U.L</u> : $t=\infty$ $u=\infty-a$ $u=\infty$

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$$= \int_0^\infty e^{-s(a+u)} \cdot f(u)\,du$$

$$= e^{-sa} \int_0^\infty e^{-su} \cdot f(u)\,du$$

$$= e^{-as} \int_0^\infty e^{-st} \cdot f(t)\,dt$$

$$= e^{-as} \cdot L\{f(t)\} = e^{-as} \cdot \tilde{f}(s)$$

Hence proved.

ANOTHER FORM OF THIS THEOREM:

If $L\{f(t)\} = \tilde{f}(s)$ and $a>0$, then $L\{f(t-a) \cdot h(t-a)\} = e^{-as} \cdot \tilde{f}(s)$.

15/09/23

CHANGE OF SCALE:

If $L\{f(t)\} = \tilde{f}(s)$, then $L\{f(at)\} = \frac{1}{a}\tilde{f}\left(\frac{s}{a}\right)$, (where $a>0$)

PROOF: $L\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt$.

$$= \int_0^\infty e^{-\frac{sx}{a}} \cdot f(x)\,\frac{dx}{a} = \frac{1}{a}\int_0^\infty e^{-\frac{st}{a}} \cdot f(t)\,dt$$

$$= \frac{1}{a} \cdot \tilde{f}\left(\frac{s}{a}\right).$$

Put $at = x$

$t = \frac{x}{a}$

$dt = \frac{dx}{a}$

L.L: $t=0$; U.L: $t=\infty$

$x=at$; $x=at$

$x=0$; $x=\infty$

• LAPLACE TRANSFORM OF DERIVATIVES:

If $f(t)$ is continuous and of exponential order and $f'(t)$ is sectionally continuous, then the

$$L\{f'(t)\} = s\tilde{f}(s) - f(0).$$

PROOF: $L\{f'(t)\} = \int_0^\infty \underset{u}{e^{-st}} \cdot \underset{v}{f'(t)}\,dt = \left[e^{-st} \int f'(t)\,dt\right]_0^\infty - \int_0^\infty \left[\frac{d}{dt}(e^{-st}) \int f'(t)\,dt\right]dt$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-s e^{-st}) \cdot f(t)\,dt$$

$$= [0 - 1 \cdot f(0)] + s \cdot \underbrace{L\{f(t)\}}_{\tilde{f}(s)}$$

$$= s \cdot \tilde{f}(s) - f(0)$$

Result 1: By applying the above theorem to f', we have

$$L\{f''(t)\} = s.\tilde{f}'(s) - f'(0)$$
$$= s\left[s\tilde{f}(s) - f(0)\right] - f'(0)$$
$$= s^2\tilde{f}(s) - s.f(0) - f'(0).$$

similarly,

$$L\{f^n(t)\} = s^n\tilde{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) \ldots\ldots - f^{(n)}(0).$$

Q:1) using the theorem on Laplace transforms of derivatives. Find the LT of e^{at}.

Given, $f(t) = e^{at} \Rightarrow f(0) = e^0 = 1$

$$f'(t) = a.e^{at}.$$

$$\to L\{f'(t)\} = s.\tilde{f}(s) - f(0)$$

$$L\{a.e^{at}\} = s.L\{e^{at}\} - 1$$

$$= a.L\{e^{at}\} - s.L\{e^{at}\} - 1$$

$$= (a-s)\,L\{e^{at}\} = -1$$

$$L\{e^{at}\} = \frac{\cancel{-}1}{\cancel{-}(s-a)}$$

$$L\{e^{at}\} = \frac{1}{s-a}$$

2) $f(t) = \cos at$.

$f'(t) = -a\sin at \quad \Rightarrow f(0) = \cos 0 = 1$

$f''(t) = -a^2\cos at \quad f'(0) = -a\sin 0 = 0$

$$L\{f''(t)\} = s^2\tilde{f}(s) - s.f(0) - f'(0)$$

$$L\{-a^2\cos at\} = s^2 L\{\cos at\} - s(1) - 0$$

$$-a^2 L\{\cos at\} - s^2 L\{\cos at\} = -s$$

$$\cancel{-}(s^2+a^2).L\{\cos at\} = \cancel{-}s$$

$$L\{\cos at\} = \frac{s}{s^2+a^2}$$

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3) $f(t) = t \cdot \sin at$.

$f'(t) = 1 \cdot \sin at + t(a\cos at)$ $\Rightarrow$ $f(0) = 0$

$f''(t) = a\cos at + 1\cdot(a\cos at) + t\left[-a^2 \sin at\right]$ $f'(0) = \sin 0 + 0(a\cos 0)$

$= 2a\cos at - a^2 t \sin at.$ $= 0+0 \to 0$

$\rightarrow L\{f''(t)\} = s^2 \bar{f}(s) - s \cdot f(0) - f'(0).$

$L\{2a\cos at - a^2 t \sin at\} = s^2 L\{t \sin at\} - s(0) - 0$

$2a \cdot L\{\cos at\} - a^2 \cdot L\{t \sin at\} - s^2 L\{t \sin at\} = 0.$

$\frac{2a \cdot s}{s^2+a^2} = (a^2+s^2) \cdot L\{t \cdot \sin at\}.$

$L\{t \cdot \sin at\} = \frac{2as}{(s^2+a^2)^2}$

16/09/23

- **LAPLACE TRANSFORM OF INTEGRALS:**

If $L\{f(t)\} = \bar{f}(s)$, then $L\left\{\int_0^t f(u)du\right\} = \frac{1}{s} \cdot \bar{f}(s)$

PROOF: $LHS = \int_0^\infty \left[e^{-st} \int_0^t f(u)\,du\right] dt$

$$= \left[\left(\frac{e^{-st}}{s}\right) \int_0^t f(u)du\right]_0^\infty - \int_0^\infty f(t) \cdot \frac{e^{-st}}{-s} \cdot dt$$

$$= (0-0) + \frac{1}{s}\underbrace{\int_0^\infty e^{-st} f(t)dt}_{\bar{f}(s)}$$

$$= \frac{\bar{f}(s)}{s}$$

similarly,

$$L\left\{\int_0^t \int_0^t f(u)\,du\,du\right\} = \frac{\bar{f}(s)}{s^3}$$

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Q: 1) $L\{\int_0^t e^{-t}\cos t\,dt\}$.

$$L\{\cos t\} = \frac{s}{s^2+a^2} = \frac{s}{s^2+1}$$

By FST,

$$L\{e^{-t}\cos t\} = \left[\frac{s}{s^2+1}\right]_{s\to s+1}$$

$$= \frac{s+1}{(s+1)^2+1} = \tilde{f}(s)$$

$$L\{\int_0^t e^{-t}\cos t\,dt\} = \frac{\tilde{f}(s)}{s} = \frac{1}{s}\frac{(s+1)}{(s^2+2s+2)}$$

2) Find $L\{\int_0^t\int_0^t \cosh at\,dt\,dt\}$

$$L\{\cosh at\} = \frac{s}{s^2-a^2} = \tilde{f}(s)$$

$$L\{\int_0^t\int_0^t \cosh at\,dt\,dt\} = \frac{1}{s^{\cancel{2}}}\cdot\frac{\cancel{s}}{s^2-a^2}$$

$$= \frac{1}{s(s^2-a^2)}$$

- LAPLACE TRANSFORM OF $t^n f(t)$

1. Multiplication By 't'.

Theorem : If $f(t)$ is sectionally continuous and of exponential order, and if $L\{f(t)\} = \tilde{f}(s)$, then

$$L\{t\cdot f(t)\} = -\tilde{f}'(s).$$

PROOF : We know that,

$$\tilde{f}(s) = \int_0^\infty e^{-st} f(t)\,dt ,$$

then by Leibnitz rule for differentiating under the integral sign.

$$\frac{d\tilde{f}}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt.$$

$$= \int_0^\infty (-t\cdot e^{-st})\,f(t)\,dt.$$

$$= -\int_0^\infty e^{-st}\,[t\cdot f(t)]\,dt.$$

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$= -L\{t \cdot f(t)\}$

$L\{t.f(t)\} = (-1)\tilde{f}'(s).$

In general,

$$L\{t^n f(t)\} = (-1)^n \frac{d^n \tilde{f}}{ds^n}.$$

1) $L\{t \sin at\} = \frac{-d\tilde{f}}{ds}$

$$L\{\sin at\} = \frac{a}{a^2+s^2} = \tilde{f}(s)$$

$$= \frac{-d}{ds}\left(\frac{a}{a^2+s^2}\right)$$

$$= (-1)(-1).\frac{a(1).2s}{(s^2+a^2)^2}$$

$$= \frac{2as}{(s^2+a^2)^2}$$

2) $L\{t \cdot \cos at\} = \frac{-d\tilde{f}}{ds}$

$$L\{\cos at\} = \frac{s}{s^2+a^2} = \tilde{f}(s)$$

$$= \frac{-d}{ds}\left(\frac{s}{s^2+a^2}\right) \Rightarrow \frac{vu'-uv'}{v^2}$$

$$= -\left[\frac{(s^2+a^2)\cdot 1 - s(2s+0)}{(s^2+a^2)^2}\right]$$

$$= -\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$= -\left[\frac{-(2s^2-s^2-a^2)}{(s^2+a^2)^2}\right]$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2}$$

3) $L\{t^2 \cos 3t\}$

$$L\{\cos 3t\} = \frac{s}{s^2+9} = \tilde{f}(s)$$

$$= (-1)^2 . \frac{d^2}{ds^2} * \left[\frac{s}{s^2+9}\right]$$

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$$= \frac{d}{ds}\left[\frac{(s^2+9)\cdot 1 - s(2s+0)}{(s^2+9)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{9-s^2}{(s^2+9)^2}\right]$$

$$= \frac{(s^2+9)^2(0-2s)-(9-s^2)(2(s^2+9)\cdot 2s)}{(s^2+9)^4}$$

$$= \frac{(s^2+9)^2(-2s)-(9-s^2)\,2(s^2+9)\cdot 2s}{(s^2+9)^4}$$

20/09/23

2. Division by 't'.

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds.$$

Q: 1) Find the Laplace Transform of $\frac{\sin t}{t}$.

$$L\left\{\frac{\sin t}{t}\right\}. = \frac{1}{s^2+1^2} = \bar{f}(s)$$

$$= \int_s^\infty \frac{1}{s^2+1}\cdot ds. \qquad * \left[\int_s^\infty \bar{f}(s)\,ds\right]$$

$$= \left[\tan^{-1} s\right]_s^\infty \qquad * \left[\int \frac{1}{1+s^2} = \tan^{-1}(s)\right]$$

$$= \tan^{-1}\infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

2) $L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}$

$$L\{e^{-at}-e^{-bt}\}$$

$$= \frac{1}{s+a} - \frac{1}{s+b} = \bar{f}(s)$$

$$= \int_s^\infty \bar{f}(s)\,ds \Rightarrow \int_s^\infty \left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds.$$

$$= \left[\log(s+a) - \log(s+b)\right]_s^\infty = \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$$

$$= \log\left[\frac{\cancel{s}(1+\frac{a}{s})}{\cancel{s}(1+\frac{b}{s})}\right]_s^\infty = \log\left(\frac{1+0}{1+0}\right) - \log\left(\frac{s+a}{s+b}\right)$$

$$= 0 + \log\left(\frac{s+b}{s+a}\right)$$

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3) $L\left\{\frac{\cos 2t-\cos 3t}{t}\right\}$

Exam :

$L\left\{\frac{\cos at-\cos bt}{t}\right\}$

$L\{\cos 2t-\cos 3t\}$

$= \frac{s}{s^2+4}-\frac{s}{s^2+9} = \bar{f}(s) = \int_s^\infty \bar{f}(s)\,ds.$

$= \int_s^\infty \left(\frac{s}{s^2+4}-\frac{s}{s^2+9}\right)ds.$ $\left[\int \frac{f'(x)}{f(x)}dx = \log|f(x)|\right]$

Multiply & Divide by '2'

$= \frac{1}{2}\int_s^\infty \left(\frac{2s}{s^2+4}-\frac{2s}{s^2+9}\right)ds.$

$= \frac{1}{2}\left[\log(s^2+4)-\log(s^2+9)\right]_s^\infty$

$= \frac{1}{2}\left[\log\left(\frac{s^2+4}{s^2+9}\right)\right]_s^\infty$

$= \frac{1}{2}\log\left[\frac{\cancel{s^2}(1+\frac{4}{s^2})}{\cancel{s^2}(1+\frac{9}{s^2})}\right]_s^\infty.$

$= \frac{1}{2}\left[\log\left(\frac{1+0}{1+0}\right)^{0}-\log\left(\frac{s^2+4}{s^2+9}\right)\right]$

$= \frac{1}{2}\log\left(\frac{s^2+9}{s^2+4}\right)$

4) $L\left\{\frac{\cos at-\cos bt}{t}\right\}$

$= L\{\cos at-\cos bt\}$

$= \frac{s}{s^2+a^2}-\frac{s}{s^2+b^2} = \bar{f}(s)$

$= \int_s^\infty\left(\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right)ds.$

Mult. & Div by 2

$= \frac{1}{2}\left[\log(s^2+a^2)-\log(s^2+b^2)\right]_s^\infty$

$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$

$= \frac{1}{2}\left[\log\left[\frac{\cancel{s^2}(1+\frac{a^2}{s^2})}{\cancel{s^2}(1+\frac{b^2}{s^2})}\right]\right]_s^\infty$

$= \frac{1}{2}\left[\log\left(\frac{1+0}{1+0}\right)^{0}-\log\frac{s^2+a^2}{s^2+b^2}\right]$

$= \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$

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5) $L\left\{\frac{1-\cos at}{t}\right\}$

$L\{1-\cos at\}$

$= \frac{1}{s} - \frac{s}{s^2+a^2} = \bar{f}(s)$

$= L\left\{\frac{1-\cos at}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+a^2}\right) ds.$

$= \frac{1}{2}\left[\log s^2\right]_s^\infty - \frac{1}{2}\left[\log(s^2+a^2)\right]_s^\infty$

$\cancel{\boxed{\Rightarrow [\log \infty - \log s] - \frac{1}{2}}}$

$= \frac{1}{2}\left[\log\left(\frac{s^2}{s^2+a^2}\right)\right]_s^\infty$

$= \frac{1}{2}\log\left[\frac{\cancel{s^2}}{\cancel{s^2}\left(1+\frac{a^2}{s^2}\right)}\right]_s^\infty$

$= \frac{1}{2}\left[\log\left(\cancel{\frac{1}{1+0}}\right) - \log\left(\frac{s^2}{s^2+a^2}\right)\right]$

$= \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2}\right)$

21/09/23.

→ EVALUATION OF INTEGRALS BY LAPLACE TRANSFORMS.

Sometimes evaluation of improper integrals can be done easily by using L.T technique.

$\bar{f}(s) = L\{f(t)\}$

$= \int_0^\infty e^{-st} f(t)\, dt.$

1) $\int_0^\infty t \cdot e^{-3t}\, dt.$

We note that the given integral is same as $\int_0^\infty e^{-st} \cdot t\, dt$ with $s = 3$.

$= L\{t\}$

$= \frac{1}{s^2}$

$= \frac{1}{9}$

2) $\int_0^{\infty} e^{-4t} \sin 3t \, dt$

$= L\{\sin 3t\}$ with $s=4$

$= \frac{a}{s^2+a^2}$, $s=4$

$= \frac{3}{4^2+3^2}$

$= \frac{3}{25}$ //

★★

3) $\int_0^{\infty} \frac{\sin 2t}{t} dt$. $s=0$

We note that the given integral is same as $\int_0^{\infty} e^{-st} \frac{\sin 2t}{t}$ with $s=0$

$= L\left\{\frac{\sin 2t}{t}\right\}$; $L\{\sin 2t\} = \frac{2}{s^2+4} = \bar{f}(s)$

$= \int_s^{\infty} \bar{f}(s).ds$.

$= \int_s^{\infty} \frac{2}{s^2+4} ds = \frac{\cancel{2}}{\cancel{2}} \left[\tan^{-1}\left(\frac{s}{2}\right)\right]_s^{\infty}$

$= \tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{2}\right)$

$= \frac{\pi}{2} - \tan^{-1} 0$, since $s=0$.

$= \frac{\pi}{2} - 0$

$= \frac{\pi}{2}$ //

4) $\int_0^{\infty} t^2 e^{-4t} \sin 2t \, dt = \int_0^{\infty} e^{-4t} (t^2 \sin 2t) dt$.

We note that the given integral is same as $\int_0^{\infty} e^{-st} (t^2 \sin 2t) dt$ where $s=4$.

$= L\{\sin 2t\}$. ; $L\{\sin 2t\} = \frac{a}{s^2+a^2} = \frac{2}{s^2+4} = \bar{f}(s)$

$= (-1)^2 . \frac{d^2 \bar{f}}{ds^2} = 1 . \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{2}{s^2+4}\right)\right] = \frac{d}{ds}\left[\frac{2(-1)}{(s^2+4)^2} . 2s\right]$

$= -4 . \frac{d}{ds}\left[\frac{s}{(s^2+4)^2}\right]$

$= \frac{vu' - uv'}{v^2}$

$$= -4\left[\frac{(s^2+4)^2 \cdot \frac{d}{ds}(s) - s \cdot \frac{d}{ds}\left[(s^2+4)^2\right]}{(s^2+4)^4}\right]$$

$$= -4\left[\frac{(s^2+4)^2 \cdot 1 - s \cdot 2(s^2+4) \cdot (2s+0)}{(s^2+4)^4}\right]$$

$$= \frac{-4\left[400 - 8(20)(8)\right]}{400 \times 400}$$

$$= \frac{31}{100}$$

5) $\int_0^\infty t\, e^{-3t} \sin t \, dt = \int_0^\infty e^{-3t}(t \sin t)\, dt$

We note that the given interval is same as $\int_0^\infty e^{-st}(t \sin t)\, dt$ with $s=3$

$$= L\{t \sin t\} \quad ; \quad L\{\sin t\} = \frac{1}{s^2+1} = \bar{f}(s)$$

$$\left[\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}\right]$$

$$= (-1) \cdot \frac{d\bar{f}}{ds} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= -\frac{-1}{(s^2+1)^2} \cdot 2s$$

$$= \frac{+2(3)}{(3^2+1)^2} = \frac{6}{100} = \frac{3}{50}$$

6) $\int_0^\infty e^{-3t} \cdot \cos t \, dt$

We note that the given interval is same as $\int_0^\infty e^{-st} \cos t\, dt$ with $s=3$.

$$L\{\cos t\} = \frac{s}{s^2+a^2} = \frac{s}{s^2+1} = \bar{f}(s)$$

$$= \frac{3}{9+1}$$

$$= \frac{3}{10}$$

22/09/23

FORMULAE :

8) $L\{e^{at} \cdot f(t)\} = \tilde{f}(s-a)$ (FST)

9) $L\{f(t-a) \cdot H(t-a)\} = e^{-as} \tilde{f}(s)$ (SST)

10) $L\{f(at)\} = \frac{1}{a} \tilde{f}\left(\frac{s}{a}\right)$ (change of scale)

11) $L\{f'(t)\} = s \cdot \tilde{f}(s) - f(0)$ (LT of derivatives)

12) $L\{f''(t)\} = s^2 \tilde{f}(s) - s \cdot f(0) - f'(0)$

13) a) $L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s} \tilde{f}(s)$ (LT of integrals) (Multiplication by t)

13) b) $L\{t^2 f(t)\} = (-1)^2 \tilde{f}''(s)$

14) $L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} \tilde{f}(s)\,ds.$

→ LT OF A PERIODIC FUNCTION.

A function $f(t)$ is said to be periodic if $f(t+T) = f(t)$ for some 'T' and every value of 't'.

Eg: 1) $\sin(360 + x) = \sin x$, period $= 2\pi$.

2) $\tan(\pi + x) = \tan x$, period $= \pi$

3) $|\sin(\pi + x)| = |-\sin x|$

$= |\sin x|$

period $= \pi$.

Q) $L\{e^{-2t} \cdot t^4\} = \left[\frac{4!}{s^5}\right]_{s \to s+2}$

$= \frac{24}{(s+2)^5}$

$L\{\sqrt{t}\} = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}}$

$= \frac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}} = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$

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Theorem: If $f(t)$ is periodic function with period 'T' then

$$L\{f(t)\} = \frac{1}{(1-e^{-sT})} \int_0^T e^{-st} f(t)\,dt.$$

Proof: By def, $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt.$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} + \cdots$$

Put $t = u+T$ in second integral.

$t = u+2T$ in third integral ..... so on.

$$L\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_0^T e^{-s(u+T)} f(u+T)\,du + \int_0^T e^{-s(u+2T)} f(u+2T)\,du + \cdots$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-su} f(u)\,du + e^{-2sT}\int_0^T e^{-su} f(u)\,du + \cdots$$

$$= (1+e^{-sT}+e^{-2sT}+\cdots)\int_0^T e^{-st} f(t)\,dt.$$

$$= \boxed{\frac{1}{(1-e^{-sT})} \int_0^T e^{-st} f(t)\,dt.}$$

** since $S_\infty = \frac{a}{1-r}$ (G.P) $= 1+r+r^2+\cdots\infty$.

Q: 1) Find the LT of $|\sin t|$.

$|\sin t|$ is periodic function with period 'π'

$$= \frac{1}{1-e^{-s\pi}} \int_0^\pi e^{-st} |\sin t|\,dt.$$

$$= \left(\frac{1}{1-e^{-\pi s}}\right) \int_0^\pi e^{-st}.\sin t\,dt.$$

$$= \frac{1}{(1-e^{-\pi s})} \left[\frac{e^{-st}}{s^2+1}(-s.\sin t - 1.\cos t)\right]_0^\pi$$

$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx - b\cos bx]\right]$

$$= \frac{1}{(1-e^{-\pi s})}\left[\frac{e^{-\pi s}}{s^2+1}[0-1(-1)] - \frac{e^0}{s^2+1}[0-1]\right]$$

$$= \frac{1}{(1-e^{-\pi s})}\left[\frac{e^{-\pi s}}{s^2+1} + \frac{1}{s^2+1}\right]$$

$\left[\because \sin 0 = 0, \cos 0 = 1 \quad \sin\pi = 0, \cos\pi = (-1)\right]$

23/09/23

→ INITIAL VALUE THEOREM :

**

Let $f(t)$ be continuous for all $t \geq 0$ and be of exponential order as $s \to \infty$ and if $f'(t)$ is of class A then $\underset{t\to 0}{Lt}\, f(t) = \underset{s\to\infty}{Lt}\, s \cdot L\{f(t)\}$

5M

Proof :

limit of the function value is zero

Since $f'(t)$ is sectionally continuous and of exponential order, $\underset{s\to\infty}{Lt} \int_0^\infty e^{-st} f'(t)\, dt = 0$

$$L\{f'(t)\} = s \cdot \bar{f}(s) - f(0)$$

$$\int_0^\infty e^{-st} f'(t)\, dt = s \cdot \bar{f}(s) - f(0)$$

Apply $\underset{s\to\infty}{Lt}$ on B.S.

$$\underset{s\to\infty}{Lt} \int_0^\infty e^{-st} f'(t)\, dt = \underset{s\to\infty}{Lt}\; s \cdot \underset{L\{f(t)\}}{\bar{f}(s)} - f(0)$$

$$0 = s \cdot \underset{s\to\infty}{Lt}\; L\{f(t)\} - f(0)$$

$$\boxed{f(0) = \underset{s\to\infty}{Lt}\; s \cdot L\{f(t)\}}$$

Similar to, $\boxed{\underset{t\to 0}{Lt}\, f(t) = \underset{s\to\infty}{Lt}\; s \cdot L\{f(t)\}}$ //

→ FINAL VALUE THEOREM :

Let $f(t)$ be continuous for all $t \geq 0$ and be of exponential order as $t \to \infty$ and if $f'(t)$ is of class A, then,

$$\underset{t\to\infty}{Lt}\, f(t) = \underset{s\to 0}{Lt}\; s \cdot L\{f(t)\}$$

**

Proof :

$$L\{f'(t)\} = s \cdot \bar{f}(s) - f(0).$$

$$\int_0^\infty e^{-st} f'(t)\, dt = s \cdot \bar{f}(s) - f(0)$$

Apply $\underset{s\to 0}{Lt}$ on B.S.

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put $s=0$

$$\underset{s\to 0}{Lt}\int_0^{\infty} e^{-st} f'(t)\,dt = \underset{s\to 0}{Lt}\left[s\cdot\bar{f}(s) - f(0)\right]$$

$$\int_0^{\infty} 1\cdot f'(t)\,dt = \underset{s\to 0}{Lt}\, s\cdot\bar{f}(s) - f(0).$$

$$\left[f(t)\right]_0^{\infty} = \underset{s\to 0}{Lt}\, s\cdot\bar{f}(s) - f(0).$$

$$\underset{t\to\infty}{Lt} f(t) - \cancel{f(0)} = \underset{s\to 0}{Lt}\, s\cdot\bar{f}(s) - \cancel{f(0)}.$$

$$\boxed{\underset{t\to\infty}{Lt} f(t) = \underset{s\to 0}{Lt}\, s\cdot\underbrace{\bar{f}(s)}_{L\{f(t)\}}}$$

→ A FUNCTION OF CLASS 'A'.

A function which is piecewise continuous on every finite interval in the range $t\geq 0$ and is of exponential order, as $t\to\infty$ is known as function of class A.

3M

Q: obtain the LT of $f(t)=\begin{cases}(t-1)^2 & \text{if } t>1\\ 0 & \text{if } 0<t<1.\end{cases}$

$$L\{f(t)\} = \int_0^1 0\cdot dt + \int_1^{\infty} e^{-st} f(t)\,dt$$

$$= 0 + \int_1^{\infty} \underbrace{e^{-st}}_{v}\cdot\underbrace{(t-1)^2}_{u}\,dt$$

$(t-1)^2 \quad 2(t-1) \quad 2(1-0)$ — 2 Derivatives of $(t-1)^2$

$\searrow +\quad \searrow -\quad \searrow +$

$\dfrac{e^{-st}}{-s} \quad \dfrac{e^{-st}}{(-s)^2} \quad \dfrac{e^{-st}}{(-s)^3}$

$$L\{f(t)\} = \left[\frac{(t-1)^2\cdot e^{-st}}{-s} - \frac{2(t-1)e^{-st}}{s^2} + \frac{2e^{-st}}{-s^3}\right]_1^{\infty}$$

$$= (0-0-0) - \left(0-0-\frac{2e^{-s(1)}}{s^3}\right)$$

$$= \frac{2e^{-s}}{s^3}$$

This is of the form, $g(t) = \begin{cases} f(t-a) & \text{if } t>a\\ 0 & \text{if } t<a.\end{cases}$

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Using SST, we get $L\{g(t)\} = e^{-as} \cdot \tilde{f}(s)$.

Here $a = 1$, and $f(t) = t^2$; (Remove a=1 in question.)

$$L\{f(t)\} = L\{t^2\}$$
$$= \frac{2!}{s^3} = \tilde{f}(s)$$
$$L\{g(t)\} = e^{-as} \cdot \tilde{f}(s)$$
$$= e^{-s} \cdot \frac{2!}{s^3} \Rightarrow e^{-s} \cdot \frac{2}{s^3}$$

Q: Find the LT of $(t-2)^3 \cdot u(t-2)$ $\Rightarrow \begin{cases} (t-2)^3 & \text{if } t > 2 \\ 0 & \text{if } 0 < t < 2. \end{cases}$

$a = 2$, $f(t) = t^3$.

$$L\{f(t)\} = \frac{3!}{s^4} = \frac{6}{s^4}.$$
$$L\{g(t)\} = e^{-as} \cdot \tilde{f}(s).$$
$$= e^{-2s} \cdot \frac{6}{s^4}$$

Q: $g(t) = \begin{cases} \cos\left(t - \frac{\pi}{3}\right) & \text{if } t > \frac{\pi}{3} \\ 0 & \text{if } t < \frac{\pi}{3} \end{cases}$ (or) $\cos\left(t - \frac{\pi}{3}\right) \cdot H\left(t - \frac{\pi}{3}\right)$.

$a = \frac{\pi}{3}$, $f(t) = \cos t$.

$$L\{f(t)\} = \frac{s}{s^2+1^2} = \tilde{f}(s)$$
$$L\{g(t)\} = e^{-as} \cdot \tilde{f}(s).$$
$$= e^{-\frac{\pi}{3}s} \cdot \frac{s}{s^2+1}.$$

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27/09/23

MODULE-II : INVERSE LAPLACE TRANSFORMS.

If $\bar{f}(s)$ is the LT of $f(t)$, then $f(t)$ is called the inverse laplace transform of $\bar{f}(s)$ and is denoted by $L^{-1}\{\bar{f}(s)\} = f(t)$.

L^{-1} is called ILT operator.

Formulae :

$L^{-1}\left\{\frac{1}{s}\right\} = 1$

$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$

$L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{\sinh at}{a}$

$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

$L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

$L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$ $\Rightarrow$ if 'n' is +ve integer.

$= \frac{t^n}{\Gamma(n+1)}$ $\Rightarrow$ if $n+1 > 0$.

Problems :

1) $L^{-1}\left\{\frac{2s+1}{s(s+1)}\right\}$

Perform Partial fractions,

$$\frac{2s+1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$2s+1 = A(s+1) + B(s)$$

Put $s=-1$, $2(-1)+1 = 0 + B(-1)$

$-1 = -B$

$\therefore B = 1$

Put $s=0$, $0+1 = A(0+1) + B(0)$

$A = 1$.

$\Rightarrow L^{-1}\left\{\frac{1}{s} + \frac{1}{s+1}\right\}$

$= 1 + e^{-t}$

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Q: $L^{-1}\left\{\frac{2s-5}{s^2-4}\right\}$

$L^{-1}\left\{\frac{2s-5}{s^2-4} - \frac{5}{s^2-4}\right\}$

$= 2 \cdot L^{-1}\left\{\frac{s}{s^2-4}\right\} - 5 \cdot L^{-1}\left\{\frac{1}{s^2-4}\right\}$

$= 2\cosh 2t - 5 \cdot \frac{\sinh 2t}{2}$ //

~~Q: L^{-1}~~

→ FIRST SHIFTING THEOREM.

If $L^{-1}\{\tilde{f}(s)\} = f(t)$, then $L^{-1}\{\tilde{f}(s-a)\} = e^{at} \cdot f(t)$.

Proof : We know that,

$L\{e^{at} f(t)\} = \tilde{f}(s-a)$.

$e^{at} f(t) = L^{-1}\{\tilde{f}(s-a)\}$

→ SECOND SHIFTING THEOREM.

If $L^{-1}\{\tilde{f}(s)\} = f(t)$, then $L^{-1}\{e^{-as} \cdot \tilde{f}(s)\} = G(t)$, where

$$G(t) = \begin{cases} f(t-a) & \text{if } t>a. \\ 0 & \text{if } t<a. \end{cases}$$

Proof : $L\{G(t)\} = \int_0^\infty e^{-st} G(t)\, dt$.

$= \int_0^a 0\, dt + \int_a^\infty f(t-a) \cdot e^{-st}\, dt.$

Put $t-a = x$
$t = a+x$
$dt = 0 + dx$
L.L: $t = a$; U.L: $t = \infty$
$a+x = a$; $x = \infty$
$x = 0$

$= 0 + \int_0^\infty e^{-s(a+x)} \cdot f(x)\, dx$

$= e^{-sa} \int_0^\infty e^{-sx} \cdot f(x)\, dx.$

$= e^{-as} \int_0^\infty e^{-at} f(t)\, dt$

$= e^{-as} L\{f(t)\}$

$L\{G(t)\} = e^{-as} \cdot \tilde{f}(s)$

$G(t) = L^{-1}\{e^{-as} \cdot \tilde{f}(s)\}$

$$L^{-1}\{e^{-as} \cdot \tilde{f}(s)\} = \begin{cases} f(t-a) & \text{if } t>a \\ 0 & \text{if } t<a. \end{cases}$$

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$\bar{L}^1\{\bar{e}^{as}.\tilde{f}(s)\} = f(t-a).H(t-a).$

Q: $\bar{L}^1\left\{\dfrac{\bar{e}^{3s}}{(s-4)^2}\right\}$

$= f(t-a).H(t-a)$, where $a=3$.

$= f(t-3).H(t-3)$ —— (1)

Now we find,

$\bar{L}^1\left\{\dfrac{1}{(s-4)^2}\right\} = e^{4t}.\bar{L}^1\left\{\dfrac{1}{s^2}\right\} \to$ By FST.

$\Rightarrow$ If $(s-4)\to s$, then multiply by e^{4t}.

$= e^{-4t}.\dfrac{t^1}{1!} = t e^{-4t} \Rightarrow f(t)$

$f(t-3) = (t-3).e^{-4(t-3)}$

$= (t-3).e^{-4t+12}$

$\bar{L}^1\left\{\dfrac{e^{-3s}}{(s-4)^2}\right\} = f(t-3).H(t-3)$

$= (t-3).e^{-4t+12}.H(t-3)$

$= \begin{cases} (t-3).\bar{e}^{4t+12} & \text{if } t>3. \\ 0 & \text{if } t<3 \end{cases}$

04/10/23

→ CHANGE OF SCALE PROPERTY.

If $L\{f(t)\} = \tilde{f}(s)$, then $\bar{L}^1\{\tilde{f}(as)\} = \frac{1}{a}.f\left(\frac{t}{a}\right)$ $(a>0)$.

Proof: By def,

$\tilde{f}(s) = L\{f(t)\} = \int_0^\infty \bar{e}^{st}.f(t)\,dt$

$\tilde{f}(as) = \int_0^\infty \bar{e}^{ast}.f(t)\,dt.$

put $at = x$.
$t = \frac{x}{a}$
$dt = \frac{dx}{a}$.

$= \int_0^\infty \bar{e}^{xs}.f\left(\frac{x}{a}\right).\frac{dx}{a}.$

Replace 'x' as 't'.

$= \int_0^\infty \bar{e}^{st}.f\left(\frac{t}{a}\right).\frac{dt}{a}.$

$= \frac{1}{a}\int_0^\infty \bar{e}^{st} f\left(\frac{t}{a}\right).dt. = \frac{1}{a}.L\left\{f\left(\frac{t}{a}\right)\right\}$

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$$= L\left\{\frac{1}{a} f\left(\frac{t}{a}\right)\right\}$$

$$L^{-1}\left\{\bar{f}(as)\right\} = \frac{1}{a} f\left(\frac{t}{a}\right)$$

Q: If $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \cancel{t}\cdot\frac{\sin t}{\cancel{t}}$. $\frac{t \cdot \sin t}{2} = f(t)$. Then find $L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\}$ To be proved

$$L^{-1}\left\{\frac{2s}{[(2s)^2+1]^2}\right\} = \frac{1}{a} \cdot f\left(\frac{t}{a}\right)$$

$$= \frac{1}{2} \cdot f\left(\frac{t}{2}\right) \Rightarrow \frac{1}{2} \cdot \frac{\left(\frac{t}{2}\right) \cdot \sin\left(\frac{t}{2}\right)}{2} = \frac{t}{8} \sin\left(\frac{t}{2}\right)$$

Multiply by '4' on both sides.

$$L^{-1}\left\{\frac{8s}{[4s^2+1]^2}\right\} = \cancel{4} \cdot \frac{t}{\cancel{8}_2} \cdot \sin\left(\frac{t}{2}\right)$$

$$L^{-1}\left\{\frac{8s}{[4s^2+1]^2}\right\} = \frac{t}{2} \sin\left(\frac{t}{2}\right).$$

→ INVERSE LAPLACE TRANSFORM OF INTEGRALS.

If $L^{-1}\{\bar{f}(s)\} = f(t)$, then $L^{-1}\left\{\int_s^\infty \bar{f}(s)\, ds\right\} = \frac{f(t)}{t}$.

Proof : We know that,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\, ds.$$

provided the integral exist.

$$\therefore L^{-1}\left\{\int_s^\infty \bar{f}(s)\, ds\right\} = \frac{f(t)}{t}$$

$$t \cdot L^{-1}\left\{\int_s^\infty \bar{f}(s)\, ds\right\} = f(t) = L^{-1}\left\{\bar{f}(s)\right\}.$$

Q: $L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$ $\bar{f}(s)$

$$L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = t \cdot L^{-1}\left\{\int_s^\infty \bar{f}(s)\, ds\right\}.$$

$$= t \cdot L^{-1}\left\{\int_s^\infty \frac{1}{(s+1)^2}\, ds\right\}$$

$$= t \cdot L^{-1}\left\{\left[-\frac{1}{(s+1)}\right]_s^\infty\right\}$$

$$= t \cdot L^{-1}\left\{-\frac{1}{\infty} + \frac{1}{s+1}\right\} = t \cdot L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= t \cdot e^{-t}.$$

* $\int \frac{1}{x^2}\, dx$

$\int x^{-2}\, dx$.

$-\frac{x^{-1}}{1} = -\frac{1}{x}$.

* $L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

Q: $L^{-1}\left\{\frac{2}{(s-a)^3}\right\} = t\cdot L^{-1}\left\{\int_s^\infty \frac{2}{(s-a)^3}\cdot ds\right\}$.

$\int \frac{1}{x^3}\,dx$

$\int x^{-3}\,dx$

$\frac{x^{-2}}{-2} = \frac{-1}{2x^2}$.

$= t\cdot L^{-1}\left\{2\cdot\left(\frac{-1}{2x^2}\right)_s^\infty\right\}$

$= t\cdot L^{-1}\left\{\left[\frac{\cancel{2}(-1)}{\cancel{2}(s-a)^2}\right]_s^\infty\right\}$

$= t\cdot L^{-1}\left\{-\frac{1}{\infty}+\frac{1}{(s-a)^2}\right\}$

$= t\cdot L^{-1}\left\{0+\frac{1}{(s-a)^2}\right\}$

$= t\cdot e^{at}\, L^{-1}\left\{\frac{1}{s^2}\right\}$ by FST.

$= t\cdot e^{at}\cdot\frac{t^1}{1!} \Rightarrow t^2\cdot e^{at}$

Q: $L^{-1}\left\{\frac{1}{(s-3)^4}\right\} = t\cdot L^{-1}\left\{\int_s^\infty \frac{1}{(s-3)^4}\,ds\right\}$

$\int \frac{1}{x^4}\,dx$

$\int x^{-4}\,dx$

$\frac{x^{-3}}{-3} = -\frac{1}{3x^3}$.

$= t\cdot L^{-1}\left\{\left[-\frac{1}{3(s-3)^3}\right]_s^\infty\right\}$

$= t\cdot L^{-1}\left\{-\cancel{\frac{1}{\infty}}^{0}+\frac{1}{3(s-3)^3}\right\}$.

$= t\cdot L^{-1}\left\{0+\frac{1}{3(s-3)^3}\right\}$.

$= t\cdot e^{3t}\cdot L^{-1}\left\{\frac{1}{3s^3}\right\}$ by FST.

$= \frac{t\cdot e^{3t}}{3}\left(\frac{t^2}{2!}\right)$

$= \frac{t^3\cdot e^{3t}}{3\times 2} \Rightarrow \frac{t^3\cdot e^{3t}}{6}$

Q: $L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\} = t\cdot L^{-1}\left\{\int_s^\infty \frac{s+2}{(s^2+4s+5)^2}\,ds\right\}$

Divide by Multiply by 2

$= \frac{t}{2}\cdot L^{-1}\left\{\int_s^\infty \frac{2s+4}{(s^2+4s+5)^2}\,ds\right\}$.

$\int \frac{f'(x)}{f(x)^n} = \frac{f(x)^{n+1}}{n+1}$

$= \frac{t}{2}\, L^{-1}\left\{\left[\frac{(s^2+4s+5)^{-1}}{-1}\right]_s^\infty\right\}$

$= \frac{t}{2}\, L^{-1}\left\{\left[-\frac{1}{s^2+4s+5}\right]_s^\infty\right\}$

$= \frac{t}{2}\, L^{-1}\left\{-\cancel{\frac{1}{\infty}}^{0}+\frac{1}{s^2+4s+5}\right\}$

$L^{-1}\left\{\frac{1}{s^2}\right\}+L^{-1}\left\{\frac{1}{4s}\right\}+L^{-1}\left\{\frac{1}{5}\right\}$

$t + \frac{1}{4} +$

$= \frac{t}{2}\, L^{-1}\left\{\frac{1}{s^2+4s+5}\right\}$

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** Imp

$\tan^{-1}$ when log is there in question, apply derivative.

→ INVERSE LAPLACE TRANSFORM OF DERIVATIVES.

Theorem: If $L^{-1}\{\bar{f}(s)\} = f(t)$, then $L^{-1}\{\bar{f}^n(s)\} = (-1)^n \cdot t^n \cdot f(t)$

Proof: We know that,

$$L\{t^n f(t)\} = (-1)^n \cdot \frac{d^n}{ds^n}[\bar{f}(s)].$$

$$L^{-1}\{\bar{f}^n(s)\} = (-1)^n \cdot t^n f(t).$$

Q: Find the {inverse laplace transform of} $L^{-1}\{\log(1+\frac{1}{s^2})\}$.

$$\bar{f}(s) = \log\left(\frac{s^2+1}{s^2}\right)$$

$$= \log(s^2+1) - \log s^2$$

$$= \log(s^2+1) - 2\log s.$$

$$\bar{f}'(s) = \frac{1}{(s^2+1)} \cdot 2s - 2 \cdot \frac{1}{s}.$$

Apply L^{-1} on B.S

$$L^{-1}\{\bar{f}'(s)\} = 2 \cdot L^{-1}\left\{\frac{s}{s^2+1} - \frac{1}{s}\right\}$$

Apply ILT of Derivatives,

$$(-1)^1 t^1 f(t) = 2[\cos t - 1]$$

$$-t \cdot f(t) = -2(1-\cos t)$$

$$f(t) = \frac{2(1-\cos t)}{t}$$

$$L^{-1}\{\bar{f}(s)\} = \frac{2(1-\cos t)}{t}$$

Q: $\bar{f}(s) = \log\frac{(s+3)}{s+4}$

$$= \log(s+3) - \log(s+4)$$

$$\bar{f}'(s) = \frac{1}{(s+3)} - \frac{1}{(s+4)}$$

Apply L^{-1} on B.S

$$L^{-1}\{\bar{f}'(s)\} = L^{-1}\left\{\frac{1}{s+3} - \frac{1}{s+4}\right\}$$

Apply derivative

$$(-1) t^1 f(t) = [e^{-3t} - e^{-4t}].$$

$$-t f(t) = e^{-3t} - e^{-4t}$$

$$f(t) = \frac{e^{-3t} - e^{-4t}}{-t} \Rightarrow \frac{e^{-4t} - e^{-3t}}{t}$$

$$L^{-1}\{\bar{f}(s)\} = \frac{e^{-4t} - e^{-3t}}{t}$$

Imp

Q: Find $\bar{L}^1\left\{\frac{s}{(s^2+a^2)^2}\right\}$

Proof: We know that,

$$\bar{L}^1\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a} = f(t).$$

Here, $\bar{f}(s) = \frac{1}{s^2+a^2}$.

$D\left(\frac{1}{x}\right) = \frac{-1}{x^2}$.

$$\bar{f}'(s) = -\frac{1}{(s^2+a^2)^2}(2s+0)$$

$\boxed{\bar{L}^1\{\bar{f}^n(s)\} = (-1)^n t^n f(t)}$

$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

Apply derivative,

$$\bar{L}^1\{\bar{f}'(s)\} = (-1)^1 \cdot t^1 \cdot f(t).$$

$$\bar{L}^1\left\{\frac{-2s}{(s^2+a^2)^2}\right\} = -t\cdot f(t).$$

$$2\cdot\bar{L}^1\left\{\frac{s}{(s^2+a^2)^2}\right\} = t\cdot f(t).$$

Substitute $f(t)$.

$$\bar{L}^1\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t}{2}\cdot\frac{\sin at}{a}.$$

$$\boxed{\bar{L}^1\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t\cdot\sin at}{2a}}$$

Q: $\bar{L}^1\left\{\log\frac{s^2+1}{(s-1)^2}\right\}$

$$\bar{f}(s) = \log\left(\frac{s^2+1}{(s-1)^2}\right)$$

$$= \log(s^2+1) - \log(s-1)^2 \Rightarrow \log(s^2+1) - 2\log(s-1)$$

Apply derivative,

$$\bar{f}'(s) = \frac{1}{s^2+1}\cdot 2s - 2\cdot\frac{1}{(s-1)}.$$

$$= 2\left[\frac{s}{s^2+1} - \frac{1}{s-1}\right]$$

Apply $\bar{L}^1$ on B.S.

$$\bar{L}^1\{\bar{f}'(s)\} = 2\bar{L}^1\left\{\frac{s}{s^2+1} - \frac{1}{s-1}\right\}.$$

$$(-1)t^1\cdot f(t) = 2\left[\cos t - e^t\right]$$

$$-t\,f(t) = -2\left[e^t - \cos t\right]$$

$$f(t) = \frac{2\left[e^t - \cos t\right]}{t}$$

Imp:

Q: $\bar{f}(s) = \tan^{-1}\left(\frac{a}{s}\right) + \cot^{-1}\left(\frac{s}{b}\right)$

$$\bar{f}'(s) = \frac{1}{1+\frac{a^2}{s^2}} \cdot \frac{d}{ds}\left(\frac{a}{s}\right) - \frac{1}{1+\frac{s^2}{b^2}} \cdot \frac{d}{ds}\left(\frac{s}{b}\right)$$

$$= \left(\frac{\cancel{s^2}}{s^2+a^2}\right) \cdot a\left(-\frac{1}{\cancel{s^2}}\right) - \frac{b^{\cancel{2}}}{b^2+s^2}\left(\frac{1}{\cancel{b}}\right)$$

$$= \frac{-a}{s^2+a^2} - \frac{b}{s^2+b^2}.$$

$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$

$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$

$\frac{d}{dx}(x) = 1$

Applying L^{-1} on B.S

$$L^{-1}\{\bar{f}'(s)\} = L^{-1}\left\{\frac{-a}{s^2+a^2} - \frac{b}{s^2+b^2}\right\}$$

$$(-1) \cdot t \cdot f(t) = -\sin at - \sin bt$$

$$\cancel{-}\, t f(t) = \cancel{-}(\sin at + \sin bt)$$

$$f(t) = \frac{\sin at + \sin bt}{t}$$

$$L^{-1}\{\bar{f}(s)\} = \frac{\sin at + \sin bt}{t}$$

06/10/23

→ MULTIPLICATION BY POWERS OF 's'.

Theorem: If $L^{-1}\{\bar{f}(s)\} = f(t)$ and $f(0) = 0$, then $L^{-1}\{s\bar{f}(s)\} = f'(t)$

Proof: We know that,

$$L\{f'(t)\} = s \cdot \bar{f}(s) - f(0).$$

$$L\{f'(t)\} = s \cdot \bar{f}(s) - 0$$

$$f'(t) = L^{-1}\{s \cdot \bar{f}(s)\}.$$

In general,

$$L^{-1}\{s^n \bar{f}(s)\} = f^{(n)}(t) \text{ if } f^n(0) \text{ for } n = 1, 2, 3 \ldots (n-1)$$

Q: $L^{-1}\left\{\frac{s}{2s^2-1}\right\}$

$$L^{-1}\left\{\frac{s}{2s^2-1}\right\} = L^{-1}\left\{s \cdot \frac{1}{2s^2-1}\right\}$$

First we find,

$$L^{-1}\left\{\frac{1}{2s^2-1}\right\} = L^{-1}\left\{\frac{1}{2\left(s^2-\frac{1}{2}\right)}\right\} = \frac{1}{2} L^{-1}\left\{\frac{1}{s^2-\left(\frac{1}{\sqrt{2}}\right)^2}\right\}$$

$$= \frac{1}{2} \cdot \frac{\sinh\left(\frac{1}{\sqrt{2}}t\right)}{1/\sqrt{2}}$$

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$= \frac{1}{\sqrt{2}} \sinh\left(\frac{t}{\sqrt{2}}\right) = f(t)$

$L^{-1}\left\{\frac{s}{2s^2-1}\right\} = f'(t)$. | $f(0)=0$

$= \frac{1}{\sqrt{2}} \cdot \cosh\left(\frac{t}{\sqrt{2}}\right) \cdot \frac{1}{\sqrt{2}}$

$= \frac{1}{2} \cosh\left(\frac{t}{\sqrt{2}}\right)$ //

Q: $L^{-1}\left\{\frac{s}{(s-4)^5}\right\}$

$L^{-1}\left\{\frac{s}{(s-4)^5}\right\} = L^{-1}\left\{s \cdot \frac{1}{(s-4)^5}\right\}$.

We know that.

$L^{-1}\left\{\frac{1}{(s-4)^5}\right\} = e^{4t} \cdot L^{-1}\left\{\frac{1}{(s)^5}\right\}$ BY FST.

$= e^{4t} \cdot \frac{t^4}{4!} = f(t)$. $\rightarrow f(0) = 0$.

$L^{-1}\left\{\frac{s}{(s-4)^5}\right\} = f'(t)$.

$\frac{d}{dx}(u \cdot v) = uv' + vu'$ //

$L^{-1}\left\{s \cdot \frac{1}{(s-4)^5}\right\} = \frac{d}{dt}\left[\frac{e^{4t} \cdot t^4}{24}\right]$

$= \frac{1}{24}\left[e^{4t} \cdot (4t^3) + 4e^{4t} \cdot t^4\right]$

$= \frac{4e^{4t}}{24} t^3 (1+t)$ //

$\rightarrow$ DIVISION BY 's'.

Theorem: If $L^{-1}\{\bar{f}(s)\} = f(t)$, then $L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(u)\,du$.

Proof: We know that,

$L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\bar{f}(s)$.

$L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(u)\,du$.

NOTE: $L^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\} = \int_0^t \int_0^t f(u)\,du\,du$.

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→ ~~DIVISION BY 's'.~~

Q: Find $\bar{L}^1\left\{\frac{1}{s(s+2)}\right\}$

First we find.

$$\bar{L}^1\left\{\frac{1}{s+2}\right\} = e^{-2t} = f(t)$$

$$\bar{L}^1\left\{\frac{1}{s(s+2)}\right\} = \int_0^t f(t)\,dt.$$

$$= \int_0^t e^{-2t}\,dt = \left[\frac{e^{-2t}}{-2}\right]_0^t$$

$$= -\frac{e^{-2t}}{2} + \frac{e^0}{2}$$

$$= \frac{1-e^{-2t}}{2}$$

Q: $\bar{L}^1\left\{\frac{1}{s(s^2+a^2)}\right\}$

First we find,

$$\bar{L}^1\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a} = f(t)$$

$$\bar{L}^1\left\{\frac{1}{s(s^2+a^2)}\right\} = \int_0^t f(t)\,dt = \int_0^t \frac{\sin at}{a}\,dt$$

$$= \left[\frac{-\cos at}{a^2}\right]_0^t$$

$$= \frac{-\cos at}{a^2} + \frac{\overset{1}{\cancel{\cos 0}}}{a^2}$$

$$= \frac{1-\cos at}{a^2}$$

Q: $\bar{L}^1\left\{\frac{1}{s(s^2+2s+2)}\right\}$

First we find,

$$\bar{L}^1\left\{\frac{1}{s^2+2s+2}\right\} = \bar{L}^1\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= e^{-t}.\bar{L}^1\left\{\frac{1}{s^2+1}\right\} \text{ BY FST.}$$

$$= e^{-t}.\sin t = f(t)$$

$$\bar{L}^1\left\{\frac{1}{s(s^2+2s+2)}\right\} = \int_0^t e^{-t}.\sin t\,dt.$$

$$= \left\{\frac{e^{-t}}{1^2+1^2}[-\sin t - 1.\cos t]\right\}_0^t$$

$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]\right]$

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$$= \left[-\frac{e^{-t}}{2} (\sin t + \cos t) \right]_0^t$$

$$= -\frac{e^{-t}}{2} (\sin t + \cos t) + \frac{e^0}{2}(0+1)$$

$$= \frac{1 - e^{-t}(\sin t + \cos t)}{2} \quad //$$

10M

* CONVOLUTION :

Let $f(t)$ and $g(t)$ be two functions defined for $t>0$. We definite the convolution of $f(t)$ and $g(t)$.

$$f(t) * g(t) = \int_0^t f(u) \cdot g(t-u)\, du.$$

Assuming that the integral exists.

CONVOLUTION THEOREM :

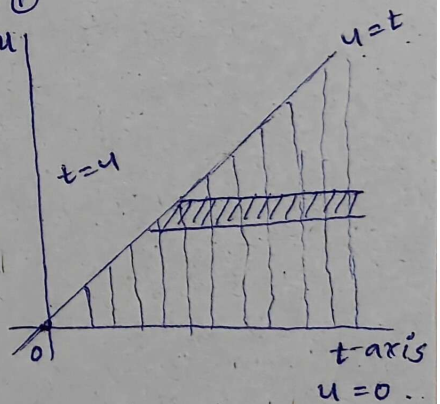
If $L^{-1}\{\tilde{f}(s)\} = f(t)$ and $L^{-1}\{\tilde{g}(s)\} = g(t)$ then,

$$L^{-1}\{\tilde{f}(s) \cdot \tilde{g}(s)\} = f(t) * g(t)$$

$$= \int_0^t f(u)\, g(t-u)\, du.$$

Proof :

$$L\{f(t) * g(t)\} = \int_0^\infty e^{-st} \left[\int_0^t f(u)\, g(t-u)\, du \right] dt. \quad \text{—} ①$$



This region lies between $u=0$ and $u=t$. To change the order of integration we consider horizontal strip through the region which enters at P. Where $t=u$ and such strips can be drawn above the line $u=0$. Hence the double integral in ① after reversing, order of integration can be written as,

$$L\{f(t) * g(t)\} = \int_0^\infty \int_u^\infty e^{-st} f(u)\, g(t-u)\, dt\, du$$

Put $(t-u) = p$.

$dt - 0 = dp$

| $t-p=$
| $L.L = u-u = 0$
| $U.L = \infty - u = \infty$.

$$L\{f(t) * g(t)\} = \int_0^\infty \int_0^\infty e^{-s(u+p)} f(u)\, g(p)\, dp\, du$$

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$$= \int_0^\infty \int_0^\infty e^{-su} e^{-sp} f(u) g(p)\, dp\, du.$$

$$= \left[\int_0^\infty e^{-su} f(u) du \right] \left[\int_0^\infty e^{-sp} g(p) dp \right]$$

$$L\{f(t) * g(t)\} = \tilde{f}(s) . \tilde{g}(s),$$

Q: Using convolution theorem, find $\cancel{L^{-1}\left\{\frac{1}{s+a}\right\}}$ $L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\}$

$$L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\} = L^{-1}\left\{\frac{1}{(s+a)} \cdot \frac{1}{(s+b)}\right\} = L^{-1}\{\tilde{f}(s) . \tilde{g}(s)\}$$

Here, $\tilde{f}(s) = \frac{1}{s+a}$ and $\tilde{g}(s) = \frac{1}{s+b}$.

$$f(t) = L^{-1}\left\{\frac{1}{s+a}\right\} \quad ; \quad g(t) = L^{-1}\left\{\frac{1}{s+b}\right\}$$

$$= e^{-at}. \qquad\qquad = e^{-bt}$$

By convolution theorem,

$$L^{-1}\{\tilde{f}(s) . \tilde{g}(s)\} = f(t) * g(t)$$

$$f(t) * g(t) = \int_0^t f(u) g(t-u) du$$

$$= \int_0^t e^{-au} . e^{-b(t-u)} . du.$$

$$= \int_0^t e^{-au} . e^{-bt} . e^{bu} . du$$

$$= e^{-bt} \int_0^t e^{(b-a)u} du.$$

$$= e^{-bt} \left[\frac{e^{(b-a)u}}{(b-a)} \right]_0^t$$

$$= \frac{e^{-bt}}{(b-a)} \left[e^{(b-a)t} - e^{0} \right]$$

$$= \frac{1}{(b-a)} \left[e^{\cancel{-bt}+\cancel{bt}-at} - e^{-bt} \right]$$

$$= \frac{e^{-at} - e^{-bt}}{b-a}$$

Put $a=2, b=3$.

Q: $L^{-1}\left\{\frac{1}{(s+2)(s+3)}\right\} = \frac{e^{-2t} - e^{-3t}}{3-2} = e^{-2t} - e^{-3t}$

07-10-23

Q: $L^{-1}\left\{\frac{1}{s(s^2+4)}\right\}$

$$L^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+2^2}\right\} = L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\}$$

Here, $\bar{g}(s) = \frac{1}{s}$ and $\bar{f}(s) = \frac{1}{s^2+2^2}$.

$f(t) = \frac{1}{s^2+4}$; $g(t) = 1$

$g(t-u) = 1$

By convolution theorem,

$$L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = f(t) * g(t)$$

$$f(t) * g(t) = \int_0^t f(u)\cdot g(t-u)\cdot du.$$

$$= \int_0^t \frac{\sin 2u}{2}\cdot 1\, du.$$

$$= \left[-\frac{\cos 2u}{2(2)}\right]_0^t$$

$$= -\frac{\cos 2t}{4} + \frac{\cos 0}{4}.$$

$$= \frac{1-\cos 2t}{4}$$

Q: $L^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\}$

$$L^{-1}\left\{\frac{1}{s((s^2+1)+1)}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s^2+2s+2)}\cdot\frac{1}{s}\right\}$$

$$= L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\}$$

Let $\bar{f}(s) = \frac{1}{(s^2+1)+1}$ and $\bar{g}(s) = \frac{1}{s}$.

$f(t) = L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$; $g(t) = L^{-1}\left\{\frac{1}{s}\right\}$

$= 1$

$= e^{-t}\cdot L^{-1}\left\{\frac{1}{s^2+1}\right\}$ BY FST $\quad g(t-u) = 1$

$= e^{-t}\cdot \sin t$

By convolution theorem,

$$L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = f(t) * g(t). = \int_0^t f(u)\cdot g(t-u)\,du;$$

$$= \int_0^t e^{-u}\sin u\cdot 1\, du.$$

$$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]\right]$$

$a=-1, b=1$

$$= \left[\frac{e^{-u}}{1+1}(-\sin u - 1\cdot\cos u)\right]_0^t$$

$$= \frac{e^{-t}}{2}(-\sin t - \cos t) - \frac{e^{0}}{2}(-0-1)$$

$$= \frac{e^{-t}}{2}(-\sin t - \cos t) + \frac{1}{2}$$

Using convolution Theorem,

Q: $L^{-1}\left\{ \frac{1}{(s^2+a^2)^2} \right\}$

$$= L^{-1}\left\{ \frac{1}{(s^2+a^2)} \cdot \frac{1}{(s^2+a^2)} \right\}$$

$$= L^{-1}\{ \bar{f}(s) \cdot \bar{g}(s) \}$$

Let $\bar{f}(s) = \frac{1}{s^2+a^2}$; $\bar{g}(s) = \frac{1}{s^2+a^2}$

$f(t) = L^{-1}\left\{ \frac{1}{s^2+a^2} \right\}$ $\quad g(t) = L^{-1}\left\{ \frac{1}{s^2+a^2} \right\}$

$f(t) = \frac{\sin at}{a}$ $\quad g(t) = \frac{\sin at}{a}$

$g(t-u) = \frac{\sin a(t-u)}{a}$

using convolution theorem,

$$L^{-1}\{ \bar{f}(s) \cdot \bar{g}(s) \} = f(t) * g(t) = \int_0^t f(u) \cdot g(t-u)\, du$$

$$= \int_0^t \frac{\sin au}{a} \cdot \frac{\sin a(t-u)}{a} \cdot du$$

$$= \left[-\frac{\cos au}{a^2} - \frac{\cos a(t-u)}{a^2} \right]_0^t$$

$$= \left[\left(-\frac{\cos at}{a^2} - \frac{\cos a(t-u)}{a^2} \right) - \left(\frac{-1}{a^2}\right.\right.$$

$$= \int_0^t \frac{2}{2} \frac{\sin \overset{A}{au}}{a} \cdot \frac{\sin(\overset{B}{at-au})}{a} \cdot du$$

$[\because 2\sin A \sin B = \cos(A-B) - \cos(A+B)]$

$2\sin au \sin(at-au)$

$= \cos(au - at + au) - \cos(au + at - au)$

$= \cos(2au - at) - \cos(at)$

$$= \int_0^t \frac{\cos(2au-at) - \overset{const}{\cos at}}{2a^2} \cdot du$$

$$= \frac{1}{2a^2}\left[\frac{\sin(2au-at)}{(2a-0)} - u\cos at \right]_0^t$$

$$= \frac{1}{2a^2}\left[\frac{\sin(2at-at)}{2a} - t\cos at \right] - \frac{1}{2a^2}\left[\frac{\sin(0-at)}{2a} - 0 \right]$$

$$= \frac{\sin at}{4a^3} - \frac{t\cos at}{2a^2} + \frac{\sin at}{4a^3}$$

$$= \frac{2\sin at}{4a^3} - \frac{t\cos at}{2a^2}$$

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Q: $L^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\}$

$= L^{-1}\left\{\frac{1}{(s^2+a^2)}\cdot\frac{1}{(s^2+b^2)}\right\}$

$= L^{-1}\left\{\bar{f}(s)\cdot\bar{g}(s)\right\}$

Let $\bar{f}(s) = \frac{1}{s^2+a^2}$; $\bar{g}(s) = \frac{1}{s^2+b^2}$.

$f(t) = L^{-1}\left\{\frac{1}{s^2+a^2}\right\}$ $\quad g(t) = L^{-1}\left\{\frac{1}{s^2+b^2}\right\}$

$f(t) = \frac{\sin at}{a}$ $\quad g(t) = \frac{\sin bt}{b}$

$g(t-u) = \frac{\sin b(t-u)}{b}$

Using convolution theorem,

$L^{-1}\left\{\bar{f}(s)\cdot\bar{g}(s)\right\} = f(t) * g(t) = \int_0^t f(u)\cdot g(t-u)\,du.$

$= \int_0^t \frac{2}{2}\,\frac{\sin au}{a}\cdot\frac{\sin(bt-bu)}{b}\,du$

$\left[\because\ 2\sin A\sin B = \cos(A-B)-\cos(A+B)\right]$

$2\sin au\cdot\sin(bt-bu)$

$= \cos(au-bt+bu) - \cos(au+bt-bu)$

$= \cos\left[(a+b)u-bt\right] - \cos\left[(a-b)u+bt\right]$

$= \int_0^t \left\{\frac{[\cos(a+b)u-bt] - \cos[(a-b)u+bt]}{2ab}\right\}du$

$= \frac{1}{2ab}\left[\frac{\sin(a+b)u-bt}{(a+b)\cdot 1} - \frac{\sin[(a-b)u+bt]}{(a-b)}\right]_0^t$

$= \frac{1}{2ab}\left[\frac{\sin(at+\cancel{bt}-\cancel{bt})}{a+b} - \frac{\sin(at-\cancel{bt}+\cancel{bt})}{a-b}\right] - \frac{1}{2ab}\left[\frac{\sin(-bt)}{a+b} - \frac{\sin bt}{a-b}\right]$

$= \frac{1}{2ab}\left[\frac{\sin at}{a+b} - \frac{\sin at}{a-b}\right] - \frac{1}{2ab}\left[\frac{\sin(-bt)}{a+b} - \frac{\sin bt}{a-b}\right]$

11-10-23.

Imp Problems (10M)

Formulae :

1) $L\{y''\} = s^2\cdot L\{y\} - s\,y(0) - 1\cdot y'(0).$

2) $L\{y'\} = s\cdot L\{y\} - y(0).$

3) $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$

4) $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

5) $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t}{2a}\sin at.$

1) Using LT Method, solve $(D^2+1)y = 6\cos 2t$. $(t>0)$

$y=3$ and $Dy=1$ when $t=0$.

Given, $y(0)=3$, $y'(0)=1$.

$$y''+y = 6\cos 2t.$$

Taking Laplace transform of this equation,

$$L\{y''\}+L\{y\} = L\{6\cos 2t\}$$

~~(Taking)~~ $s^2.L\{y\} - s.y(0) - 1.y'(0) + L\{y\} = 6.L\{\cos 2t\}$

$$s^2.L\{y\} - s(3) - 1(1) + L\{y\} = 6\left(\frac{s}{s^2+4}\right)$$

$$(s^2+1)L\{y\} = 3s+1+\frac{6s}{s^2+4}.$$

$$L\{y\} = \frac{3s}{s^2+1} + \frac{1}{s^2+1} + \frac{6s}{(s^2+1)(s^2+4)} \quad \text{——} \ 1$$

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{\text{difference}}\left[\frac{1}{\text{smaller}} - \frac{1}{\text{larger}}\right]$$

$$= \frac{1}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right] \quad (4-1)$$

$$L\{y\} = \frac{3s}{s^2+1} + \frac{1}{s^2+1} + \cancel{6}^{2}s.\frac{1}{\cancel{3}}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right]$$

Taking ILT on ~~B~~.S, we get,

$$y = 3.L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\} + 2.L^{-1}\left\{\frac{s}{s^2+1} - \frac{s}{s^2+4}\right\}.$$

$$= 3\cos t + \sin t + 2(\cos t - \cos 2t)$$

$$= 3\cos t + \sin t + 2\cos t - 2\cos 2t$$

$$y = 5\cos t + \sin t - 2\cos 2t.$$

2) Using LT Method, solve the equation, $\frac{d^2y}{dt^2} + 2.\frac{dy}{dt} + 5y = e^{-t}\sin t$

given that $y(0)=0$, $y'(0)=1$.

$$y(0)=0, \quad y'(0)=1$$

Given, $y''+2y'+5y = e^{-t}\sin t$.

Taking LT of this equation.

$$L\{y''\} + 2.L\{y'\} + 5L\{y\} = L\{e^{-t}\sin t\}.$$

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$s^2 \cdot L\{y\} - s \cdot y(0) - 1 \cdot y'(0) + 2[s \cdot L\{y\} - 1 \cdot y(0)] + 5 \cdot L\{y\} = L\{e^{-t} \cdot \sin t\}$

$s^2 \cdot L\{y\} - s \cdot y(0) - 1 \cdot y'(0) + 2 \cdot s \cdot L\{y\} - 2(0) + 5 \cdot L\{y\} = L\{e^{-t} \sin t\}$

$s^2 \cdot L\{y\} - s(0) - 1(1) + 2 \cdot s \cdot L\{y\} - 2(0) + 5 \cdot L\{y\}$

$s^2 \cdot L\{y\} - s(0) - 1(1) + 2 \cdot s \cdot L\{y\} - 0 + 5L\{y\} = \frac{1}{(s+1)^2+1}$ by FST $\left[L\{\sin t\} = \frac{1}{s^2+1},\ \frac{1}{(s+1)^2+1} = e^{-t} \sin t\right]$

$s^2 \cdot L\{y\} - 0 - 1 + 2s \cdot L\{y\}$

$(s^2+2s+5)\, L\{y\} = 1 + \frac{1}{s^2+2s+2}$

$$L\{y\} = \frac{1}{s^2+2s+5} + \frac{1}{(s^2+2s+5)(s^2+2s+2)}. \quad \text{—— (1)}$$

Put $(s^2+2s) = x$

$$\frac{1}{(s^2+2s+5)(s^2+2s+2)} = \frac{1}{(x+5)(x+2)} = \frac{1}{\text{difference}}\left[\frac{1}{\text{smaller}} - \frac{1}{\text{larger}}\right]$$

$$= \frac{1}{3}\left[\frac{1}{x+2} - \frac{1}{x+5}\right]$$

$$= \frac{1}{3}\left[\frac{1}{s^2+2s+2} - \frac{1}{s^2+2s+5}\right]$$

(1) becomes,

$$L\{y\} = \frac{1}{s^2+2s+5} + \frac{1}{3}\left[\frac{1}{s^2+2s+2} - \frac{1}{s^2+2s+5}\right].$$

Taking L^{-1} on B.S,

$$y = L^{-1}\left\{\frac{1}{(s+1)^2+4} + \frac{1}{3} \cdot \frac{1}{(s+1)^2+1} - \frac{1}{3} \cdot \frac{1}{(s+1)^2+4}\right\}$$

Replacing $s+1$ by s and using FST.

$$y = e^{-t} \cdot L^{-1}\left\{\frac{1}{s^2+4} + \frac{1}{3(s^2+1)} - \frac{1}{3(s^2+4)}\right\}$$

$$= e^{-t}\left[\frac{\sin 2t}{2} + \frac{1}{3} \cdot \sin t - \frac{1}{3} \frac{\sin 2t}{2}\right]$$

$$= e^{-t}\left[\left(1 - \frac{1}{3}\right) \cdot \frac{\sin 2t}{2} + \frac{\sin t}{3}\right]$$

$$= e^{-t}\left[\frac{\cancel{2}}{3} \cdot \frac{\sin 2t}{\cancel{2}} + \frac{\sin t}{3}\right]$$

$$= e^{-t}\left[\frac{\sin 2t}{3} + \frac{\sin t}{3}\right]$$

$$= e^{-t}\left[\frac{2 \sin 2t}{3}\right],$$

$L^{-1}\left\{e^{-2s}(3+5s)\right\}$

$f(t-a) \cdot H(t-a)$; $a = 2$

$f(t-2) \cdot H(t-2)$ —— (1)

$f(t-2) \cdot e^{-2(t-2)}$

$f(t-2) = (t-2) \cdot e^{-2(t-2)} (t-2) \cdot u(t-2)$

$L^{-1}\{3+5s\}$

$3t+5$

A) $L^{-1}\left\{\frac{3+5p}{p^2 e^{2p}}\right\}$ $\frac{(3+5p)\, e^{-2p}}{p^2} \Rightarrow \frac{(3+5s)\, e^{-2s}}{s^2} =$

$$\frac{e^{-2s}\ 3+5s}{s^2}$$

$e^{-2s} t \cdot (3+5s)$

$L^{-1}\left\{e^{-2s}\ \frac{3+5s}{s^2}\right\}$

$L^{-1}\left\{\frac{3+5s}{s^2}\right\} = \frac{3}{s^2} + \frac{5s}{s^2}$

$3L^{-1}\left\{\frac{1}{s^2}\right\} + 5\left\{\frac{s}{s^2}\right\}$

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19/10/23

10 M PROBLEMS.

Solve using LT Method :

1) $\frac{d^2x}{dt^2} + 2 \cdot \frac{dx}{dt} + x = 3te^{-t}$, given $x(0) = 4$, $\frac{dx}{dt} = 0$ at $t = 0$.

(OR)

$\frac{d^2y}{dt^2} + 2 \cdot \frac{dy}{dt} + y = 3te^{-t}$, given $y(0) = 4$, $y'(0) = 0$.

$y'' + 2y' + y = 3te^{-t}$.

Taking 'Laplace' transform' of this equation.

$L\{y''\} + 2 \cdot L\{y'\} + L\{y\} = L\{3te^{-t}\}$.

$s^2 L\{y\} - s \cdot y(0) - 1 \cdot y'(0) + 2[s \cdot L\{y\} - 1 \cdot y(0)] + L\{y\} = 3 \cdot L\{e^{-t} \cdot t\}$

$s^2 L\{y\} - s(4) - 0 + 2s \cdot L\{y\} - 2(4) + L\{y\} = 3\left[\frac{1}{s^2}\right]_{s \to s+1}$ BY FST.

$(s^2 + 2s + 1) L\{y\} - 4s - 8 = 3 \cdot \frac{1}{(s+1)^2}$

$(s+1)^2 + L\{y\} = 4s + 8 + \frac{3}{(s+1)^2}$.

$L\{y\} = \frac{4s}{(s+1)^2} + \frac{8}{(s+1)^2} + \frac{3}{(s+1)^4}$

$y = L^{-1}\left\{\frac{4(s+1-1)}{(s+1)^2} + \frac{8}{(s+1)^2} + \frac{3}{(s+1)^4}\right\} \rightarrow s+1 \rightarrow e^{-t}$.
$s+1 \rightarrow s$.

$= e^{-t} \cdot L^{-1}\left\{\frac{4(s-1)}{s^2} + \frac{8}{s^2} + \frac{3}{s^4}\right\}$ BY FST,

$y = e^{-t} \cdot L^{-1}\left\{\frac{4s}{s^2} - \frac{4}{s^2} + \frac{8}{s^2} + \frac{3}{s^4}\right\}$

$= e^{-t} \cdot L^{-1}\left\{\frac{4}{s} - \frac{4}{s^2} + \frac{8}{s^2} + \frac{3}{s^4}\right\}$

$= e^{-t} \cdot L^{-1}\left\{\frac{4}{s} + \frac{4}{s^2} + \frac{3}{s^4}\right\}$

$= e^{-t}\left[4(1) + 4\left(\frac{t^1}{1!}\right) + 3\left(\frac{t^3}{3!}\right)\right]$

$y = e^{-t}\left[4 + 4t + \cancel{3}\frac{t^3}{\cancel{6}_2}\right]$

$y = e^{-t}\left[4 + 4t + \frac{t^3}{2}\right]$,,

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2) solve the initial value problem by using LT Method.

$$y'' + 7y' + 10y = 4e^{-3t}, \quad y(0) = 0, \quad y'(0) = -1.$$

$$y'' + 7y' + 10y = 4e^{-3t}.$$

Taking LT on both sides,

$$L\{y''\} + 7L\{y'\} + 10L\{y\} = L\{4e^{-3t}\}$$

$$s^2 L\{y\} - s \cdot y(0) - y'(0) + 7[s \cdot L\{y\} - s \cdot y(0)] + 10L\{y\} = 4 \cdot \frac{1}{(s+3)}$$

$$s^2 L\{y\} - s(0) - (-1) + 7s \cdot L\{y\} - s(0) + 10L\{y\} = \frac{4}{s+3}.$$

$$(s^2 + 7s + 10) L\{y\} - 0 + 1 - 0 = \frac{4}{s+3}.$$

$$(s+5)(s+2) L\{y\} = \frac{4}{s+3} - 1$$

$$L\{y\} = \frac{4}{(s+3)(s+5)(s+2)} - \frac{1}{(s+5)(s+2)}$$

Partial fractions:

$$\frac{1}{(s+3)(s+5)(s+2)} = \frac{A}{s+3} + \frac{B}{s+5} + \frac{C}{s+2}$$

put $s = -3$, $A = \frac{1}{(-3+5)(-3+2)} = \frac{-1}{2}$

put $s = -5$, $A = \frac{1}{(-5+3)(-5+2)} = \frac{1}{6}$

put $s = -2$, $A = \frac{1}{(-2+3)(-2+5)} = \frac{1}{3}$

$$\frac{1}{(s+5)(s+2)} = \frac{1}{\text{diff}}\left[\frac{1}{\text{smaller}} - \frac{1}{\text{larger}}\right] = \frac{1}{3}\left[\frac{1}{(s+2)} - \frac{1}{(s+5)}\right]$$

$$L\{y\} = 4\left[-\frac{1}{2(s+3)} + \frac{1}{6(s+5)} + \frac{1}{3(s+2)}\right] - \frac{1}{3}\left[\frac{1}{(s+2)} - \frac{1}{(s+5)}\right]$$

$$= \frac{-\cancel{4}^{2}}{\cancel{2}(s+3)} + \frac{\cancel{4}^{2}}{\cancel{6}_{3}(s+5)} + \frac{4}{3(s+2)} - \frac{1}{3(s+2)} + \frac{1}{3(s+5)}.$$

$$= \frac{-2}{s+3} + \left(\frac{2}{3} + \frac{1}{3}\right)\left(\frac{1}{s+5}\right) + \left(\frac{4}{3} - \frac{1}{3}\right) \cdot \frac{1}{s+2}.$$

$$y = L^{-1}\left\{\frac{-2}{s+3} + \frac{1}{s+5} + \frac{1}{s+2}\right\}$$

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$y = -2e^{-3t} + e^{-5t} + \cancel{2}e^{-2t}$

$y = -2e^{-3t} + e^{-5t} + e^{-2t}$ //

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12-10-23 MODULE - III. FOURIER TRANSFORMS.

A transformation is mathematical operation which converts one function into another. The integral transform of a function $f(x)$ is defined by,

$$\tilde{f}(s) = \int_a^b f(x) \cdot K(s,x)\, dx \quad \text{---(1)}, \text{ where 's' is parameter.}$$

$K(s,x)$ is called kernel of the transforms.

If a & b are finite, then, the integral transform is said to be finite transform. Otherwise it is said to be infinite transform.

(i) If $K(s,x) = e^{isx}$, then (1) is called fourier transform.

(ii) If $K(s,x) = \sin x$, " " " " Fourier sine transform.

(iii) " " $= \cos x$, " " " " Fourier cosine transform

→ DIRICHLET'S CONDITION :

A function $f(x)$ is said to satisfy Dirichlet's conditions in an interval (a,b) if :

(i) $f(x)$ is defined and is single valued except possibly at finite number of points in the interval a,b and,

(ii) $f(x)$ and $f'(x)$ are piecewise continuous in the interval (a,b).

At a point of discontinuity,

$$f(x) = \frac{1}{2}\left[f(x+0) + f(x-0)\right].$$

A periodic function $f(x)$ is defined in $(-l,l)$ can be expressed in fourier series by extending this concept, non-perodic functions defined as interval $(-\infty,\infty)$ can be expressed as fourier integral.

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→ FOURIER INTEGRAL THEOREM.

Let $f(x)$ be a function which satisfies Dirichlet's conditions in function $(-l, l)$. Then $f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\, d\lambda$.

Let us assume that 'l' increases indefinitely we may write $\frac{n\pi}{l} = \lambda$ and $\frac{\pi}{l} = d\lambda$.

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cdot\left[2\int_0^\infty \cos\lambda(t-x)\,dt\right] d\lambda.$$

Q: Using Fourier integral, show that $e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda \sin\lambda x\, d\lambda}{(\lambda^2+a^2)(\lambda^2+b^2)}$.

Since, the integrant on RHS contains 'sine' term, we use Fourier sine integral formula.

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin px \int_0^\infty f(t)\sin pt\, dt\, dp.$$

Replacing 'p' with 'λ',

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\infty f(t)\sin\lambda t\, dt\, d\lambda. \quad -(1)$$

Let $f(x) = e^{-ax} - e^{-bx}$.

$$f(t) = e^{-at} - e^{-bt}. \quad —— (2)$$

Substituting (2) in (1)

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\int_0^\infty (e^{-at} - e^{-bt})\sin\lambda t\, dt\right] d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x\cdot\left\{\int_0^\infty e^{-at}\sin\lambda t\, dt - \int_0^\infty e^{-bt}\sin\lambda t\, dt\right\} d\lambda.$$

$a = \lambda,\ s = a.$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x\left\{\frac{\lambda}{a^2+\lambda^2} - \frac{\lambda}{b^2+\lambda^2}\right\} d\lambda.$$

$$= \frac{2}{\pi}\int_0^\infty \lambda\sin\lambda x\left[\frac{1}{\lambda^2+a^2} - \frac{1}{\lambda^2+b^2}\right] d\lambda.$$

$$e^{-ax} - e^{-bx} = \frac{2}{\pi}\int_0^\infty \lambda\sin\lambda x\left[\frac{(\lambda^2+b^2)-(\lambda^2+a^2)}{(\lambda^2+a^2)(\lambda^2+b^2)}\right] dx.$$

$$e^{-ax} - e^{-bx} = \frac{2}{\pi}(b^2-a^2)\int_0^\infty \frac{\lambda\sin\lambda x\, dx}{(\lambda^2+a^2)(\lambda^2+b^2)}$$

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$$e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty \cos\lambda x\left[\int_0^\infty e^{-t}\cos t\cos\lambda t\,dt\right]d\lambda.$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\cos\lambda x}{2}\left[\int_0^\infty e^{-t}\cos(\lambda+1)t\,dt + \int_0^\infty e^{-t}\cos(\lambda-1)t\,dt\right]d\lambda.$$

$L\{\cos(\lambda+1)t\}$ $\quad$ $L\{\cos(\lambda-1)t\}$

$$\left[\because \cos A\cos B = \frac{2\cos\lambda t\cos t}{2} = \cos(\lambda t+t)+\cos(\lambda t-t)\right]$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\cos\lambda x}{2}\left[\frac{1}{(\lambda+1)^2+1^2}\right]+\left[\frac{1}{(\lambda-1)^2+1^2}\right]d\lambda,$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\cos\lambda x}{2}\left[\frac{1}{1+\lambda^2+2\lambda+1}+\frac{1}{1+\lambda^2+1-2\lambda}\right]d\lambda.$$

$$e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty \frac{\cos\lambda x}{2}\left[\frac{1}{\lambda^2+2+2\lambda}+\frac{1}{\lambda^2+2-2\lambda}\right]d\lambda.$$

$$= \frac{\cancel{2}}{\pi}\int_0^\infty \frac{\cos\lambda x}{\cancel{2}}\left[\frac{\lambda^2+2-\cancel{2\lambda}+\lambda^2+2+\cancel{2\lambda}}{(\lambda^2+2)^2-4\lambda^2}\right]d\lambda$$

$$= \frac{\cancel{2}}{\pi}\int_0^\infty \frac{\cos\lambda x}{\cancel{2}}\,\frac{(\lambda^2+2)2}{(\lambda^2+2)^2-4\lambda^2}\,d\lambda \longrightarrow [\lambda^4+4+\cancel{4\lambda^2}-\cancel{4\lambda^2}].$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\lambda^2+2}{\lambda^4+4}\cos\lambda x\,dx.$$

Q: Express the function $f(x)=\begin{cases}1 & \text{for } |x|\le 1\\ 0 & \text{for } |x|>1\end{cases}$ as fourier integral.

Hence evaluate $\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$. And also find $\int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda$.

Fourier integral of $f(x)$ is given by,

$$f(x) = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\,d\lambda.$$

$$= \frac{1}{\pi}\int_0^\infty\int_{-1}^{1} 1.\cos\lambda(t-x)\,dt\,d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty\left[\frac{\sin\lambda(t-x)}{\lambda(1-0)}\right]_{-1}^{1}d\lambda.$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin\lambda(1-x)-\sin\lambda(-1-x)}{\lambda}\,d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin(\lambda-\lambda x)+\sin(\lambda+\lambda x)}{\lambda}\,d\lambda.$$

$\sin(A+B)+\sin(A-B) = 2\sin A\cos B.$

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{2\sin\lambda\cos\lambda x}{\lambda}\,d\lambda.$$

$$= \int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,dx = \frac{\pi}{2}f(x). \quad\text{—(1)}$$

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$$= \begin{cases} \frac{\pi}{2}(1) & \text{for } |x| \le 1 \\ \frac{\pi}{2}(0) & \text{for } |x| > 1 \end{cases}$$

As, $f(x)$ is discontinuous at $x=1$, the integral has the value

$$\frac{\frac{\pi}{2}(1) + \frac{\pi}{2}(0)}{2} = \frac{\pi}{4}$$

Put $x=0$ in eq. 1.

$$\int_0^\infty \frac{\sin\lambda}{\lambda} \cos 0 \cdot d\lambda.$$

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

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22/11/23

MODULE - III . FOURIER SERIES.

Q: Using Fourier integral, show that $\int_0^\infty \left(\frac{1-\cos\pi\lambda}{\lambda}\right) \sin x\lambda \, d\lambda = \begin{cases} \frac{\pi}{2} & \text{if } 0<x<\pi \\ 0 & \text{if } x>\pi. \end{cases}$

Let $f(x) = \begin{cases} \frac{\pi}{2} & \text{if } 0<x<\pi \\ 0 & \text{if } x>\pi. \end{cases}$

Then $f(t) = \begin{cases} \frac{\pi}{2} & \text{if } 0<t<\pi \\ 0 & \text{if } t>\pi. \end{cases}$ — ①

We know that Fourier sine integral of $f(x)$ is,

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\infty f(t) \sin\lambda t \, dt \, d\lambda \quad \text{— ②}.$$

From ① and ②,

$$f(x) = \frac{\cancel{2}}{\cancel{\pi}}\int_0^\infty \sin\lambda x \int_0^\pi \frac{\cancel{\pi}}{\cancel{2}} \sin\lambda t \, dt \, d\lambda.$$

$$= \int_0^\infty \sin\lambda x \left[\frac{-\cos\lambda t}{\lambda}\right]_0^\pi d\lambda$$

$$= \int_0^\infty \sin\lambda x \left[\frac{-\cos\lambda\pi + \cos 0}{\lambda}\right] d\lambda.$$

$$= \int_0^\infty \sin\lambda x \left(\frac{1-\cos\lambda\pi}{\lambda}\right) d\lambda.$$

Q: Express $f(x) = \begin{cases} 1 & \text{for } 0\le x\le\pi \quad \text{— ①} \\ 0 & \text{for } x>\pi. \end{cases}$, as a fourier sine integral and hence evaluate $\int_0^\infty \frac{1-\cos(\pi\lambda)}{\lambda} \sin(x\lambda) \, d\lambda$.

Pf: Fourier sine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\infty f(t) \sin\lambda t \, dt \, d\lambda \quad \text{— ②}$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\pi 1 \cdot \sin\lambda t \, dt \, d\lambda.$$

From ① and ②

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\pi 1\cdot\sin\lambda t \, dt \, d\lambda.$$

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$$I = \frac{2}{\pi}\int_0^\infty \sin\lambda x \left[\frac{-\cos\lambda t}{\lambda}\right]_0^\pi d\lambda.$$

$$I = \frac{2}{\pi}\int_0^\infty \sin\lambda x \left[\frac{-\cos\lambda\pi + \cos 0^\circ}{\lambda}\right] d\lambda.$$

$$I = \frac{2}{\pi}\int_0^\infty \sin\lambda x \left(\frac{1-\cos\pi\lambda}{\lambda}\right) d\lambda.$$

For $x=\pi$, which is point of discontinuity of $f(x)$ the value of the integral is.

$$= \frac{\pi}{2}\left[\frac{f(\pi-0)+f(\pi+0)}{2}\right]$$

$$= \frac{\pi}{2}\left[\frac{1+0}{2}\right]$$

$$= \frac{\pi}{4}$$

Q: Find the Fourier integral of $f(x) = \begin{cases} 0 & x<0 \\ \frac{1}{2} & x=0 \\ e^{-x} & x>0. \end{cases}$

Fourier integral of $f(x)$ is,

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\lambda(t-x)\,dt\,d\lambda.$$

$$= \frac{1}{\pi}\int_0^\infty \int_0^\infty e^{-t}\cos\lambda(t-x)\,dt\,d\lambda.$$

$$\left\{\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\,\cancel{\sin}^{\cos}\, bx + b\,\cancel{\cos}^{\sin}\, bx\right]\right\}$$

Here, $a=-1,\ b=\lambda.$

$$f(x) = \frac{1}{\pi}\int_0^\infty \left\{\frac{e^{-t}}{1+\lambda^2}\left[-1\cdot\cos\lambda(t-x) + \lambda\sin(t-x)\right]\right\}_0^\infty d\lambda.$$

$$= \frac{1}{\pi}\int_0^\infty \left[0 - \frac{1}{1+\lambda^2}\left\{-\cos\lambda(0-x) + \lambda\sin\lambda(0-x)\right\}\right] d\lambda.$$

$\because \sin(-\theta) = -\sin\theta$

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$$= -\frac{1}{\pi}\int_0^{\infty} \frac{-\cos\lambda x - \lambda\sin\lambda x}{1+\lambda^2}\,d\lambda.$$

$$= \frac{1}{\pi}\int_0^{\infty} \frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\,d\lambda.$$

verification :

put $x=0$, then,

$$f(0) = \frac{1}{\pi}\int_0^{\infty} \frac{1+0}{1+\lambda^2}\,d\lambda.$$

$$= \frac{1}{\pi}\left[\tan^{-1}\lambda\right]_0^{\infty}$$

$$= \frac{1}{\pi}\left[\tan^{-1}\infty - \tan^{-1}0\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \tan^{-1}0\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{1}{\cancel{\pi}} \times \frac{\cancel{\pi}}{2}$$

$$f(0) = \frac{1}{2}$$

Therefore verified.

Fourier Transforms :

Q: Find the fourier transform of $f(x)$ defined by,

$$f(x) = \begin{cases} e^{iqx} & \alpha < x < \beta. \\ 0 & x < \alpha \text{ and } x > \beta \end{cases}$$

**

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f(x)\,dx.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\alpha}^{\beta} e^{ipx} \cdot e^{iqx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\alpha}^{\beta} e^{i(p+q)x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{i(p+q)x}}{i(p+q)}\right]_{\alpha}^{\beta}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{i(p+q)\beta} - e^{i(p+q)\alpha}}{i(p+q)}\right]$$

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23/11/23

Q: Find the Fourier transform of $f(x) = \begin{cases} a^2-x^2 & \text{for } |x|<a \\ 0 & \text{for } |x|>a \end{cases}$. UT-3

Deduce that $\int_0^\infty \frac{\sin t - t\cos t}{t^3}\,dt = \frac{\pi}{4}$.

Soln:-

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{ipx}\,dx.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2-x^2)\,e^{ipx}\,dx.$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{(a^2-x^2)e^{ipx}}{ip} + \frac{2xe^{ipx}}{(-1)p^2} - \frac{2e^{ipx}}{-ip^3}\right]_{-a}^{a}$$

$$\left[\because \int (a^2-x^2)\,\boxed{e^{ipx}} = \quad a^2-x^2 \searrow \frac{e^{ipx}}{ip} \quad 0-2x \searrow \frac{e^{ipx}}{(-1)\Leftarrow i^2p^2} \quad -2 \searrow \frac{e^{ipx}}{i^3p^3}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left(0+\frac{2ae^{ipa}}{-p^2}+\frac{2e^{ipa}}{ip^3}\right)-\left(0-\frac{2ae^{-ipa}}{-p^2}+\frac{2e^{-ipa}}{ip^3}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[-\frac{2ae^{ipa}}{p^2}-\frac{2ae^{-ipa}}{p^2}+\frac{2e^{ipa}}{ip^3}-\frac{2e^{-ipa}}{ip^3}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-2a}{p^2}\left(e^{ipa}+e^{-ipa}\right)+\frac{2}{ip^3}\left(e^{ipa}-e^{-ipa}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-2a}{p^2}\cdot 2\cos ap+\frac{2}{ip^3}\cdot 2i\sin pa\right] \qquad \left[\because e^{i\theta}+e^{-i\theta} = 2\cos\theta, \ \because e^{i\theta}-e^{-i\theta} = 2i\sin\theta\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-4ap\cos ap+4\sin ap}{p^3}\right]$$

$$= \frac{4}{\sqrt{2\pi}}\cdot\frac{(\sin ap - ap\cos ap)}{p^3} = F(p) \rightarrow \text{Fourier Transform}$$

By inverse Fourier transform,

$$\left[\because e^{-i\theta} = \cos\theta - i\sin\theta\right]$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(p)\cdot e^{-ipx}\,dp.$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{4}{\sqrt{2\pi}}\,\frac{(\sin ap - ap\cos ap)}{p^3}(\underbrace{\cos px}_{\text{Real}} - i\underbrace{\sin px}_{\text{Imaginary}})\,dp.$$

Equating real parts,

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$$f(x) = \frac{4^{2}}{2\pi} \int_{-\infty}^{\infty} \frac{(\sin ap - ap\cos ap)}{p^3} \cos px \, dp \quad -(1)$$

$$f(x) = f(0) = a^2 - x^2 = a^2 - 0 = a^2.$$

case-(i): put $x=0$, then

$$f(0) = \frac{2}{\pi} \, 2 \cdot \int_0^\infty \frac{(\sin ap - ap\cos ap)}{p^3} \cdot \overset{\cos 0^\circ}{1} \, dp$$

$$= \frac{4}{\pi} \int_0^\infty \left(\frac{\sin ap - ap\cos ap}{p^3} \right) dp = a^2.$$

$$\int_0^\infty \left(\frac{\sin ap - ap\cos ap}{p^3} \right) dp = \frac{\pi a^2}{4}$$

put $a=1$,

$$\int_0^\infty \left(\frac{\sin p - p\cos p}{p^3} \right) dp = \frac{\pi}{4}.$$

(or) $\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3} \right) dx = \frac{\pi}{4}$, (or) $\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3} \right) dx = \frac{-\pi}{4}$,

case-(ii): put $x = \frac{1}{2}$, $a=1$, in eq^n (1)

$$f\left(\tfrac{1}{2}\right) = a^2 - x^2 = 1 - \frac{1}{4} = \frac{3}{4},$$

Equation (1) becomes,

$$\frac{2}{\pi} \cdot 2 \int_0^\infty \frac{(\sin p - 1 \cdot p\cos p)}{p^3} \cos\left(p \cdot \tfrac{1}{2}\right) dp. = f\left(\tfrac{1}{2}\right)$$

$$\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3} \right) \cos\left(\frac{x}{2}\right) dx = \frac{\pi}{4} \cdot \frac{3}{4}$$
$$= \frac{3\pi}{16},$$

(or) $\int_0^\infty \frac{(x\cos x - \sin x)}{x^3} \cos\left(\frac{x}{2}\right) dx = -\frac{3\pi}{16}$,

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Q : Find the Fourier transform of

$$f(x) = \begin{cases} 1-x^2 & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1. \end{cases}$$

(i) $\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\left(\frac{x}{2}\right)dx = \frac{-3\pi}{16}.$

Solution is same as the previous solution.

24/11/23

10M

Q : Find Fourier sine and Fourier cosine transform.

(i) $f(x) = x$

$$F_s(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin px \, dx.$$

$$F_c(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos px \, dx.$$

$\rightarrow$ Let $F(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty x e^{-ipx} dx.$

put, $ipx = y$

$x = \frac{y}{ip}$

$dx = \frac{dy}{ip}.$

$\Gamma(n) = \int_0^\infty e^{-x}.x^{n-1} dx.$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{y}{ip} \cdot e^{-y} \cdot \frac{dy}{ip}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left(\frac{1}{(-p^2)}\right) \int_0^\infty x e^{-x} dx.$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{-p^2} \int_0^\infty e^{-x} \cdot x^{2-1} dx.$$

$$= \frac{1}{p^2}\sqrt{\frac{2}{\pi}} \cdot \Gamma(2) = \frac{-1}{p^2}\sqrt{\frac{2}{\pi}}$$

$$F(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty x(\cos px - i\sin px)\,dx = \frac{-1}{p^2}\sqrt{\frac{2}{\pi}}.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty (x\cos px \, dx - ix \sin px)\,dx = \frac{-1}{p^2}\sqrt{\frac{2}{\pi}} + 0.$$

Equating real,

$$F_c(p) = \frac{-1}{p^2}\sqrt{\frac{2}{\pi}}$$

$$F_s(p) = 0$$

** MID - II

Q: Find the fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$ and deduce $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \sin px \, dx = \tan^{-1}\left(\frac{p}{a}\right) - \tan^{-1}\left(\frac{p}{b}\right)$.

Sol: Fourier sine transform of $f(x)$ is,

$$F_s\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin px \, dx. \quad (= y \text{ say}).$$

$$y = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin px \, dx.$$

Differentiation w.r.to 'p'.

$$\frac{dy}{dp} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}(x \cos px)}{x} dx. \Rightarrow \int_0^\infty e^{-st} f(x)\, dx = L\{f(x)\}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+p^2} \quad \left\{\sin L\{\cos at\} = \frac{s}{s^2+a^2}\right\}$$

Integrating w.r.t to p.

$$y = \sqrt{\frac{2}{\pi}} \int \frac{a}{a^2+p^2} dp \qquad \left[\because \int \frac{1}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\cancel{a}}{\cancel{a}} \cdot \tan^{-1}\left(\frac{p}{a}\right)\right] + c.$$

If $p = 0$, then $F_s\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin 0 \, dx = 0 = y$, — (i)

Also, $y = \sqrt{\frac{2}{\pi}} \cdot \cancel{\frac{a}{a}} \tan^{-1}\left(\frac{0}{a}\right) + c$

$$y = 0 + c. \quad \text{— (ii)}$$

(i) & (ii) Equating

$$0 = 0 + c \Rightarrow c = 0.$$

$$F_s\left\{\frac{e^{-ax}}{x}\right\} = \sqrt{\frac{2}{\pi}} \left[\tan^{-1}\left(\frac{p}{a}\right)\right]$$

Illy, $F_s\left\{\frac{e^{-ax}}{x} - \frac{e^{-bx}}{x}\right\} = \sqrt{\frac{2}{\pi}}\left[\tan^{-1}\left(\frac{p}{a}\right) - \tan^{-1}\left(\frac{p}{b}\right)\right]$

i.e., $\cancel{\sqrt{\frac{2}{\pi}}} \int_0^\infty \left(\frac{e^{-ax} - e^{-bx}}{x}\right) \sin px \, dx = \cancel{\sqrt{\frac{2}{\pi}}} \left[\tan^{-1}\left(\frac{p}{a}\right) - \tan^{-1}\left(\frac{p}{b}\right)\right]$.

$$\therefore \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \sin px \, dx = \tan^{-1}\left(\frac{p}{a}\right) - \tan^{-1}\left(\frac{p}{b}\right)$$

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SEM

Q: Find the fourier cosine transform of $f(x) = \frac{1}{1+x^2}$. Hence find fourier sine transform of $g(x) = \frac{x}{1+x^2}$.

$$F_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos px \, dx.$$

Let $y = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{1+x^2} \cos px \, dx$. — (1)

Differentiating w.r. to 'p' on B.S,

$$\frac{dy}{dp} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{-1 \cdot x \sin px}{1+x^2} dx$$

Multiply & divide by x.

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x^2 \sin px}{x(1+x^2)} dx.$$

Adding and subtracting '1'.

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{(1+x^2-1)\sin px}{x(1+x^2)} dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cancel{(1+x^2)} \sin px}{x\cancel{(1+x^2)}} dx + \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin px}{x(1+x^2)} dx \quad \left[\because \int_0^\infty \frac{\sin px}{x} dx = \frac{\pi}{2}\right]$$

$$\frac{dy}{dp} = -\sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} + \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin px}{x(1+x^2)} dx \quad — (2)$$

Differentiating (2) on B.S

$$\frac{d^2y}{dp^2} = 0 + \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1 \cdot \cancel{x} \cos px}{\cancel{x}(1+x^2)} dx$$

$$D^2 y = y \rightarrow \text{from (1)}$$

$$(D^2-1)y = 0$$

A.E: $m^2 - 1 = 0$

$m = \pm 1$

$y = c_1 e^p + c_2 e^{-p}$ — (i)

$$\frac{dy}{dp} = c_1 e^p - c_2 e^{-p}$$

From (2),

$$-\sqrt{\frac{\pi}{2}} + \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin px}{x(1+x^2)} dx = c_1 e^p - c_2 e^{-p}. \quad [\because e^0 = 1]$$

put $p = 0$, $-\sqrt{\frac{\pi}{2}} + 0 = c_1 - c_2$

$\therefore c_1 - c_2 = -\sqrt{\frac{\pi}{2}}.$

From (i) i.e.,

$$y = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{1+x^2} \cos px \, dx$$

substitute 'y' from (i)

$$c_1 e^p + c_2 e^{-p} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos px}{1+x^2} dx.$$

put $p=0$,

$$c_1 + c_2 = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{1+x^2} dx.$$

$$= \sqrt{\frac{2}{\pi}} \left[\tan^{-1} x\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\tan^{-1}\infty - \tan^{-1} 0\right]$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \left(\frac{\pi}{2} - 0\right)$$

$$\therefore c_1 + c_2 = \sqrt{\frac{\pi}{2}}.$$

Now,

$$c_1 + \cancel{c_2} = \cancel{\sqrt{\frac{\pi}{2}}}$$

$$c_1 - \cancel{c_2} = -\cancel{\sqrt{\frac{\pi}{2}}}$$

$$2c_1 = 0$$

$$\therefore c_1 = 0$$

$$\Rightarrow c_1 + c_2 = \sqrt{\frac{\pi}{2}}$$

$$c_2 = \sqrt{\frac{\pi}{2}}$$

y

$$y = c_1 e^p + c_2 e^{-p}$$

$$F_c\{f(x)\} = 0 + \sqrt{\frac{\pi}{2}} e^{-p}$$

$$\frac{dy}{dp} = c_1 e^p - c_2 e^{-p}$$

$$= 0 - \sqrt{\frac{\pi}{2}} e^{-p}$$

$$\Rightarrow -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x}{1+x^2} \sin px \, dx = -\sqrt{\frac{\pi}{2}} e^{-p}$$

$$F_s\left\{\frac{x}{1+x^2}\right\} = \sqrt{\frac{\pi}{2}} \cdot e^{-p}$$

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Imp

Q: Find the fourier cosine transform of e^{-x^2}.

$$y = F_c\{e^{-x^2}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cos px \, dx. \quad \text{---}(1)$$

Differentiate w.r. to p.

$$\frac{dy}{dp} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} [-x \sin px] \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \int_0^\infty (\underset{\text{2nd}}{-2x\, e^{-x^2}}) \underset{\text{1st}}{\sin px} \, dx.$$

u.v

Calculation:

$$\int e^{-x^2}(-2x)\,dx = \int e^t \, dt = e^t = e^{-x^2}$$

let, $-x^2 = t$

$-2x\,dx = dt$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \left[\sin px \cdot e^{-x^2} - \int (p \cos px)\, e^{-x^2} dx \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \left[(0-0) - p \int_0^\infty e^{-x^2} \cos px \, dx. \right.$$

$[u.v = u\int v - \int u'(\int v)]$

$$= -\frac{p}{2} \cdot \underbrace{\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cos px \, dx.}_{y}$$

$$\frac{dy}{dp} = \frac{-py}{2}$$

Apply '$\int$' on B.S,

$$\int \frac{dy}{dp} = \int \frac{-p}{2} \, dp.$$

$$\log y = \frac{-p^2}{4} + \log c$$

$$y = c e^{-p^2/4}.$$

put $p = 0$,

then $y = ce^0$

$y = c$.

Now put $P=0$, in (1), then

$$y = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cos px \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cdot \overset{\cos 0}{1} \, dx.$$

$$\left[\because \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \right]$$

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$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{\pi}}{2}.$$

$$= \frac{1}{\sqrt{2}}$$

$$\therefore c = \frac{1}{\sqrt{2}}$$

$$y = c e^{-p^2/4}$$

$$\therefore y = \frac{1}{\sqrt{2}} e^{-p^2/4}$$

• THEOREM : If $F\{f(x)\} = F(p)$, then $F\{e^{iax} \cdot f(x)\} = F(p+a)$.

Proof : $F\{e^{iax} \cdot f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} \cdot f(x) \cdot e^{ipx} \, dx.$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(p+a)x} \cdot f(x) \, dx$$

$$= F(p+a)$$

• SHIFTING THEOREM :

If $F\{f(x)\} = F(p)$, then , $F\{f(x-a)\} = e^{ipa} \cdot F(p)$.

Proof :-

$$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} \cdot f(x-a) \, dx.$$

$\Rightarrow$ put $x - a = y$.

$x = y + a$

$dx = dy$.

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ip(y+a)} \cdot f(y) \, dy.$$

$$= e^{ipa} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipy} \cdot f(y) \, dy.$$

$$= e^{ipa} F(p)$$

• CHANGE OF SCALE PROPERTY.

~~If $F\{f(ax)\} = \frac{1}{a} F\left(\frac{p}{a}\right)$, then~~

If $F\{f(x)\} = F(p)$, then $F\{f(ax)\} = \frac{1}{a} \cdot F\left(\frac{p}{a}\right)$.

Proof !

$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f(ax) \, dx.$$

$\Rightarrow$ put $ax = y$

$x = \frac{y}{a}$

$dx = \frac{dy}{a}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ip\left(\frac{y}{a}\right)} \cdot f(y) \cdot \frac{dy}{a}.$$

$$= \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\left(\frac{p}{a}\right)x} \cdot f(x)\,dx.$$

$$= \frac{1}{a} F\left(\frac{p}{a}\right)$$

25/11/23

- MODULATION THEOREM.

If $F\{f(x)\} = F(p)$, then $F\{f(x)\cos ax\} = \frac{1}{2}\left[F(p+a) + F(p-a)\right]$.

Proof: $LHS = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot \cos ax \cdot e^{ipx}\,dx.$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left[\frac{e^{iax} + e^{-iax}}{2}\right] e^{ipx} \cdot dx.$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{i(a+p)x} + \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{i(p-a)x}\,dx.$$

$$= \frac{1}{2}\left[F(p+a) + F(p-a)\right]$$

- theorem:

If $F\{f(x)\} = F(p)$, then $F\{x^n f(x)\} = (-i)^n \cdot \frac{d^n}{dp^n}[F(p)]$

Proof: $F(p) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{ipx}\,dx$

Differentiating w.r.to p 'n' times.

$$\frac{d^n}{dp^n}[F(p)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\,(ix)^n \cdot e^{ipx}\,dx.$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[f(x) \cdot x^n\right] e^{ipx}\,dx = (-i)^n \cdot \frac{d^n}{dp^n}[F(p)]$$

~~$F\{x^n f(x)\}$~~

$$F\{x^n f(x)\} = (-i)^n \frac{d^n}{dp^n} \cdot [F(p)]$$

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Derivative
Theorem: If $F\{f(x)\} = F(p)$, then $F\{f'(x)\} = -ipF(p)$. if $f(x) \to 0$ as $x \to \pm\infty$

proof. $F\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underset{\text{2nd}}{f'(x)} \cdot \underset{\text{1st}}{e^{ipx}} dx.$

$$= \frac{1}{\sqrt{2\pi}} \left[e^{ipx} \int f'(x)\, dx - \int ip\, e^{ipx} \cdot f(x)\, dx \right]_{-\infty}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left[e^{ipx} f(x) \right]_{-\infty}^{\infty} - \frac{ip}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} \cdot f(x)\, dx.$$

$$= (0-0) - ip \cdot \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f(x) dx}_{F(p)}.$$

$$= -ipF(p).$$

similarly,

$$F\{f^{(n)}(x)\} = (-ip)^n F(p) \quad \text{if } f' \to 0.$$

Integral
Theorem: If $F\{f(x)\} = F(p)$, then $F\left\{\int_a^x f(x) dx\right\} = \frac{F(p)}{-ip}$.

proof:
Let $\phi(x) = \int_a^x f(x)\, dx.$

then $\phi'(x) = f(x)$

$$F\{\phi'(x)\} = -ip \cdot F\{\phi(x)\}$$

$$\frac{F\{f(x)\}}{-ip} = F\left\{\int_a^x f(x)\, dx\right\}$$

$$\frac{F(p)}{-ip} = F\left\{\int_a^x f(x)\, dx\right\}$$

• PROPERTIES OF SINE AND COSINE TRANSFORMS.

1) Linear property:

$$F_s\{a.f(x) + b.g(x)\} = a.F_s\{f(x)\} + b.F_s\{g(x)\}.$$

2) $F_s\{f(x) \cos ax\}$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos ax \sin px\, dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{f(x)\, 2\cos ax \sin px}{2} dx$$

$$= \frac{1}{2} \cdot \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \left[\sin(p+a)x + \sin(p-a)x\right] dx$$

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$= \frac{1}{2}\left[F_s(p+a) + F_s(p-a) \right]$

3) $F_s\{f(ax)\} = \frac{1}{a} F_s\left(\frac{p}{a}\right).$

put $ax = y$

$x = \frac{y}{a}$

$dx = \frac{dy}{a}$

$$F_s = \sqrt{\frac{2}{\pi}} \int_0^\infty f(ax) \sin px \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(y) \sin\left(\frac{p}{a}\right) y \cdot \frac{dy}{a},$$

$$= \frac{1}{a} \cdot F_s\left(\frac{p}{a}\right)$$

Q: Find the fourier sine transform of $\frac{1}{x}$.

$$F_s\left\{\frac{1}{x}\right\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x} \sin px \, dx.$$

$\left[\because \frac{\sin px}{x} = \frac{\pi}{2}\right]$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2}$$

$$= \sqrt{\frac{\pi}{2}}$$

Q: Find the fourier sine and cosine transforms of x^{n-1}, $0<n<1$ and hence show that $f(x) = \frac{1}{\sqrt{x}}$ is self-reciprocal under Fourier sine and cosine transform.

$$F_c\{x^{n-1}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{n-1} \cos px \, dx$$

We know that,

$$\int_0^\infty e^{-ax} \cdot x^{n-1} \cdot dx = \frac{\Gamma(n)}{a^n}$$

Put $a = ip$.

$$\int_0^\infty e^{-ipx} \cdot x^{n-1} \cdot dx = \frac{\Gamma(n)}{i^n p^n} = \frac{\Gamma(n)}{p^n}(-i)^n.$$

$$\int_0^\infty x^{n-1}\left[\cos px - i \sin px\right] dx = \frac{\Gamma(n)}{p^n}\left[\cos\frac{\pi}{2} - i \sin\frac{\pi}{2}\right]^n \sqrt{\frac{2}{\pi}}$$

$$= \frac{\Gamma(n)}{p^n}\left[\cos\frac{n\pi}{2} - i \sin\frac{n\pi}{2}\right] \sqrt{\frac{2}{\pi}}.$$

Equating real & imaginary parts,

$$\sqrt{\frac{2}{\pi}} \int_0^\infty x^{n-1} \cos px \, dx = \frac{\Gamma(n)}{p^n} \cos\frac{n\pi}{2}.$$

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$$F_c\{x^{n-1}\} = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n)}{p^n}\cos\frac{n\pi}{2}.$$

$$F_s\{x^{n-1}\} = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n)}{p^n}\sin\frac{n\pi}{2},$$

06/12/23

- FINITE FOURIER SINE TRANSFORM OF $F(x)$ WHEN $0<x<l$ IS DEFINED AS

$$F_s\{f(x)\} = F_s\{n\}.$$

$$= \sqrt{\frac{2}{\pi}}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

where 'n' is an integer and $f(x) = \frac{2}{l}\sum_{n=1}^{\infty} F_s(n)\sin\left(\frac{n\pi x}{l}\right)$ is called the inverse finite Fourier sine transform of $F_s(n)$.

similarly,

FINITE FOURIER COS TRANSFORM,

$$F_c(n) = \sqrt{\frac{2}{\pi}}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx.$$

Inverse formula is,

$$f(x) = \frac{1}{l}\int_0^l f(x)\,dx + \frac{2}{l}\sum_{n=1}^{\infty} F_c(n)\cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{1}{l}F_c(0) + \frac{2}{l}\sum_{n=1}^{\infty} F_c(n)\cos\left(\frac{n\pi x}{l}\right),$$

Q: Find the finite fourier sine and cosine transforms of $f(x)=1$

Range of x is not given, we take usual range $[0,\pi]$

$l=\pi$.

$$F_s(n) = \sqrt{\frac{2}{\pi}}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\pi} 1\cdot\sin\left(\frac{n\cancel{\pi}x}{\cancel{\pi}}\right)dx$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{\cos(nx)}{n}\right]_0^{\pi}$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{\cos n\pi}{n} + \frac{\cos 0}{n}\right]$$

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$$= \sqrt{\frac{2}{\pi}} \left(\frac{1-\cos n\pi}{n} \right)$$

$$\left[\because \cos n\pi = (-1)^n \right]$$

$$F_s(n) = \sqrt{\frac{2}{\pi}} \left[\frac{1-(-1)^n}{n} \right]$$

$$\rightarrow F_c(n) = \sqrt{\frac{2}{\pi}} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\pi 1 \cdot \cos\left(\frac{n\cancel{\pi} x}{\cancel{\pi}}\right) dx.$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin(nx)}{n} \right]_0^\pi$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin n\pi - \sin 0}{n} \right] = 0 \text{ for } n \neq 0.$$

If $n = 0$,

$$F_c(0) = \sqrt{\frac{2}{\pi}} \int_0^\pi 1 \cdot \overset{\cos 0}{1} \cdot dx.$$

$$= \sqrt{\frac{2}{\pi}} [x]_0^\pi$$

$$= \sqrt{\frac{2}{\pi}} (\pi - 0)$$

$$F_c(0) = \sqrt{2\pi} \,_{//}$$

Q:- Show that the finite sine transform of $\frac{x}{\pi}$ is $(-1)^{p+1} \cdot \frac{1}{p}\sqrt{\frac{2}{\pi}}$

$$F_s(p) = \sqrt{\frac{2}{\pi}} \int_0^l f(x) \sin\left(\frac{p\pi x}{l}\right) dx.$$

$$F_s\left\{\frac{x}{\pi}\right\} = \sqrt{\frac{2}{\pi}} \int_0^\pi \underset{u}{\frac{x}{\pi}} \cdot \underset{v}{\sin\left(\frac{p\cancel{\pi} x}{\cancel{\pi}}\right)} dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\overset{u}{\frac{x}{\pi}} \overset{\int v}{\left(\frac{-\cos px}{p}\right)} \overset{-\int du \cdot \int v}{-} \int_0^\pi \frac{1}{\pi} \left[-\frac{\cos px}{p} \right] dx. \right.$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cancel{\pi}}{\cancel{\pi}} \frac{\cos(p\pi)}{p} - 0 + \frac{1}{\pi p} \left[\frac{\sin px}{p} \right]_0^\pi \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{1}{p}(-1)^p + \frac{1}{\pi p^2} (\sin p\pi - \sin 0) \right. \quad \left[\because \cos p\pi = (-1)^p \right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{(-1)^{p+1}}{p} + 0.$$

$$= (-1)^{p+1} \cdot \frac{1}{p}\sqrt{\frac{2}{\pi}} \,_{//}$$

Q: Find $f(x)$ if it's finite sine transform is given by, (→ inverse)

$$F_s(P) = \frac{1-\cos P\pi}{\pi^2 P^2}, \text{ when } 0<x<\pi.$$

$$f(x) = \frac{2}{l}\sqrt{\frac{2}{\pi}} \sum_{P=1}^{\infty} F_s(P) \sin\left(\frac{P\pi x}{l}\right)$$

$$= \frac{2}{l}\sqrt{\frac{2}{\pi}} \sum_{P=1}^{\infty} F_s(P) \sin\left(\frac{P\pi x}{\pi}\right)$$

$$= \frac{2}{l}\sqrt{\frac{2}{\pi}} \sum_{P=1}^{\infty} \left(\frac{1-\cos P\pi}{\pi^2 P^2}\right) \sin Px.$$

Q: Find $f(x)$ if (→ inverse) $F_c(P) = \dfrac{6\left(\sin \frac{P\pi}{2} - \cos P\pi\right)}{(2P+1)\pi}$ for $P = 1, 2, 3, \ldots$ and $\frac{2}{\pi}$ for $p=0$, where $0<x<4$.

Soln: Given $F_c(0) = \frac{2}{\pi}$, Here $l=4$.

$$f(x) = \frac{1}{l} F_c(0) + \frac{2}{l} \sum F_c(P) \cos\left(\frac{P\pi x}{l}\right)$$

$$= \frac{1}{4}\cdot\frac{2}{\pi} + \frac{2}{4}\sum_{P=1}^{\infty} \frac{6\left(\sin\frac{P\pi}{2} - \cos P\pi\right)}{(2P+1)\pi} \cos\left(\frac{P\pi x}{4}\right).$$

Q: Find $f(x)$ if $F_c(P) = \dfrac{\cos\left(\frac{2P\pi}{3}\right)}{(2P+1)^2}$ if $0<x<1$

$$F_c(0) = \frac{\cos 0}{(0+1)^2} = \frac{1}{1} = 1,$$

Here $l=1$.

$$f(x) = \frac{1}{l} F_c(0) + \frac{2}{l} \sum F_c(P) \cos\left(\frac{P\pi x}{l}\right)$$

$$= \frac{1}{1}\cdot 1 + \frac{2}{1}\sum_{P=1}^{\infty} \frac{\cos\left(\frac{2P\pi}{3}\right)}{(2P+1)^2} \cdot \cos\left(\frac{P\pi x}{1}\right),$$

Q: Find the finite fourier cosine transform of $f(x) = \sin ax$ in $(0,\pi)$.

Here $l=\pi$

$$F_c(P) = \sqrt{\frac{2}{\pi}} \int_0^l f(x) \cos\left(\frac{P\pi x}{\pi}\right) dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\pi} \frac{2 \sin ax \cos Px}{2}\, dx.$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \left[\sin(a+P)x + \sin(a-P)x\right] dx$$

$\left[\because 2\sin A\cos B = \sin(A+B) + \sin(A-B)\right]$

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$$= \frac{1}{2}\cdot\sqrt{\frac{2}{\pi}}\left[-\frac{\cos(a+p)x}{a+p} - \frac{\cos(a-p)x}{a-p}\right]_0^{\pi}$$

$$= \frac{1}{\sqrt{2\pi}}\left[-\frac{\cos(a+p)\pi}{a+p} - \frac{\cos(a+p)\pi}{a-p} + \frac{1}{a+p} + \frac{1}{a-p}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-(-1)^{a+p}}{a+p} - \frac{(-1)^{a-p}}{a-p} + \frac{1}{a+p} + \frac{1}{a-p}\right]$$

$\because \cos(a+p)\pi = (-1)^{a+p}$

$\because \cos(a-p)\pi = (-1)^{a-p}$.

07/12/23

- CONVOLUTION.

The function,

$$H(x) = F * G.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(u).G(x-u)\,du.$$

is called the convolution or falting of two integrable functions F and G over the interval $(-\infty, \infty)$.

- CONVOLUTION THEOREM OF FOURIER TRANSFORMS.

The convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

$$F\{f(x) * g(x)\} = F\{f(x)\}.F\{g(x)\}.$$

10M * sem

Q: Use the sine inversion formula to obtain $f(x)$, if $F_s(p) = \frac{p}{1+p^2}$.

PF: Using Fourier sine inversion formula, we have.

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(p)\sin px\,dp.$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{p}{1+p^2}\sin px\,dp. \quad \text{—— } (*)$$

$\times$ & $\div$ '2'

$$y = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{p^2}{p(1+p^2)}\sin px\,dp.$$

Add & sub '1'

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{(p^2+1-1)}{p(1+p^2)} \sin px \, dp.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \left[\frac{\cancel{(p^2+1)}}{p\cancel{(1+p^2)}} - \frac{1}{p(1+p^2)}\right] \sin px \, dp.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin px}{p} dp - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin px}{p(1+p^2)} dp. \quad \left[\because \frac{\sin px}{p} = \frac{\pi}{2}\right]$$

$$y = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin px}{p(1+p^2)} dp. \quad \text{--- (1)}$$

Differentiate

$$\frac{dy}{dx} = 0 - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cancel{p}\cos px}{\cancel{p}(1+p^2)} dp \quad \text{--- (2)}$$

$$\frac{d^2y}{dx^2} = -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{-p\sin px}{1+p^2} dp.$$

$$\frac{d^2y}{dx^2} = +\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{p\sin px}{1+p^2} dp.$$

$\therefore \ d^2y = y$ – From (*).

$D^2y = y$

$(D^2-1)y = 0$

A.E: $m^2-1=0.$

$m = \pm 1$

$y = c_1e^x + c_2e^{-x}.$

$$\frac{dy}{dx} = c_1e^x - c_2e^{-x}.$$

consider,

$c_1e^x + c_2e^{-x} = y.$

$$c_1e^x + c_2e^{-x} = -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{p}{1+p^2} \sin px \, dp + \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2}. \quad \text{– from (1).}$$

put $x=0$.

$$c_1e^0 + c_2e^0 = -\sqrt{\frac{2}{\pi}} \cdot 0 + \sqrt{\frac{\pi}{2}}.$$

$$c_1 + c_2 = \sqrt{\frac{\pi}{2}}. \quad \text{--- (3)}$$

consider,

$$c_1e^x - c_2e^{-x} = \frac{dy}{dx}.$$

$$c_1e^x - c_2e^{-x} = -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos px}{1+p^2} dp \quad \text{– From (2)}$$

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put $x=0$

$$c_1 e^0 - c_2 e^0 = -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{1+p^2}\, dp.$$

$$c_1 - c_2 = -\sqrt{\frac{2}{\pi}} \left[\tan^{-1} p\right]_0^\infty$$

$$= -\sqrt{\frac{2}{\pi}} \left[\tan^{-1}\infty - \tan^{-1} 0\right]$$

$$= -\sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2} - 0\right]$$

$$c_1 - c_2 = -\sqrt{\frac{\pi}{2}}.$$

$$\rightarrow \quad c_1 - \cancel{c_2} = -\cancel{\sqrt{\frac{\pi}{2}}}$$

$$c_1 + \cancel{c_2} = \cancel{\sqrt{\frac{\pi}{2}}}$$

$$2c_1 = 0$$

$$\therefore c_1 = 0$$

$$c_1 + c_2 = \sqrt{\frac{\pi}{2}}.$$

$$0 + c_2 = \sqrt{\frac{\pi}{2}}$$

$$y = c_1 e^x + c_2 e^{-x}$$

$$= 0 + \sqrt{\frac{\pi}{2}}\, e^{-x}.$$

$$\therefore y = \sqrt{\frac{\pi}{2}}\, e^{-x} \;//$$

Q: Find $f(x)$ if its cosine transform is $F_c(p) = \begin{cases} \frac{1}{\sqrt{2\pi}}\left(a - \frac{p}{2}\right) & \text{if } p < 2a \\ 0 & \text{if } p \geq 2a \end{cases}$

using Fourier cosine inversion formula, we have

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(p) \cos px\, dp.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{2a} \frac{1}{\sqrt{2\pi}} \underbrace{\left(a - \frac{p}{2}\right)}_{u} \underbrace{\cos px}_{v}\, dp.$$

$$\rightarrow \quad \left(a - \frac{p}{2}\right) \qquad 0 - \frac{1}{2}.$$

$$(+) \qquad\qquad (-)$$

$$\frac{\sin px}{x} \qquad\qquad \frac{-\cos px}{x^2}.$$

$$f(x) = \frac{1}{\pi}\left[\left(a-\frac{p}{2}\right)\cdot\frac{\sin px}{x} - \frac{1}{2}\cdot\frac{\cos px}{x^2}\right]_0^{2a}$$

$$f(x) = \frac{1}{\pi}\left[\left(a-\frac{\cancel{2}a}{\cancel{2}}\right) - \frac{1}{2}\frac{\cos 2ax}{x^2} - 0 + \frac{1}{2}\cdot\frac{\cos 0}{x^2}\right]$$

$$f(x) = \frac{1}{\pi}\left[0 - \frac{\cos 2ax}{2x^2} + \frac{1}{2x^2}\right]$$

$$f(x) = \frac{1}{\pi}\frac{(1-\cos 2ax)}{2x^2} \text{ //}$$

Re-done.

Q: Find $f(x)$ if its sine transform is

$$F_s(p) = \frac{e^{-ap}}{p}.$$

Using Fourier sine inversion formula, we have.

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(p)\sin px\, dp.$$

$$y = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ap}}{p}\sin px\, dp.$$

$$\frac{dy}{dx} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ap}(p\cos px)}{p}\, dp. \qquad \int_0^\infty e^{-ap} \quad \left[\because L\{\cos at\} = \frac{s}{s^2+a^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}.$$

08/12/23.

• **PARSEVAL'S IDENTITY.**

If $f(x)$ is given function defined in $(-\infty, \infty)$ then it satisfies the identity,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(p)|^2 dp.$$

Q: Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ using Fourier Transform.

Let $f(x) = e^{-ax}$, $g(x) = e^{-bx}$.

Fourier cosine Transform of $e^{-ax} = F_c(e^{-ax})$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos px\, dx$$

$\downarrow$

$L\{\cos at\}$

$\downarrow$

$\frac{s}{s^2+a^2}$

$$= \sqrt{\frac{2}{\pi}}\cdot\frac{a}{a^2+p^2}$$

Here $s = a$

$a = p$.

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$$\int_0^\infty F_c[f(x)] \cdot F_c[g(x)]\, dp = \int_0^\infty f(x) \cdot g(x)\, dx.$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+p^2} \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{b}{b^2+p^2}\, dp = \int_0^\infty e^{-ax} \cdot e^{-bx}\, dx.$$

$$\frac{2ab}{\pi} \int_0^\infty \frac{1}{a^2+x^2} \cdot \frac{1}{b^2+x^2}\, dx = \int_0^\infty e^{-(a+b)x}\, dx = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$\int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)}\, dx = \frac{\pi}{2ab}\left[\frac{e^{-\infty} - e^0}{-(a+b)}\right]$$

$$= \frac{\pi}{2ab}\left[\frac{0-1}{-(a+b)}\right]$$

$$= \frac{\pi}{2ab(a+b)}$$

Redone.

Q: Find $f(x)$ if its sine transform is $F_s(p) = \frac{e^{-ap}}{p}$.

Q: Evaluate $\int_0^\infty \frac{dx}{(a^2+x^2)^2}$ using parseval's identity

Fourier cosine transform of $e^{-ax} = F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos px\, dx$

$\downarrow$ $L\{\cos at\}$ $\downarrow$ $\frac{s}{s^2+a^2}$

Here $s = a$, $a = p$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+p^2} := F(p)$$

The parseval's identity is,

$$\int_{-\infty}^\infty |f(x)|^2\, dx = \int_{-\infty}^\infty |F(p)|^2\, dp.$$

$$\int_{-\infty}^\infty |e^{-ax}|^2\, dx = \int_{-\infty}^\infty \left|\frac{\sqrt{2}}{\pi} \cdot \frac{a}{a^2+p^2}\right|^2 dp.$$

$$\frac{\cancel{2}}{\pi} \int_{-\infty}^\infty \frac{a^2}{(a^2+p^2)^2}\, dp = \cancel{2} \int_0^\infty e^{-2ax}\, dx.$$

$$\frac{1}{\pi} \cdot 2 \int_0^\infty \frac{a^2}{(a^2+x^2)^2}\, dx = \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$\frac{2a^2}{\pi} \int_0^\infty \frac{1}{(a^2+x^2)^2}\, dx = \left[\frac{e^{-\infty} - e^0}{-2a}\right]$$

$$\int_0^\infty \frac{1}{(a^2+x^2)^2}\, dx = \frac{\pi}{2a^2}\left[\frac{0-1}{-2a}\right]$$

$$= \frac{\pi}{4a^3}$$

Redone

Q: Find $f(x)$ if its sine transform is $F_s(p) = \frac{e^{-ap}}{p}$.

Using Fourier sine inversion formula we have.

$$y = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ap}}{p} \sin px \, dp.$$

$$\frac{dy}{dx} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ap}}{\cancel{p}} (\cancel{p} \cos px) \, dp.$$

$L\{\cos at\} \to \frac{s}{s^2+a^2} = \frac{a}{a^2+x^2}$

Here $s = a$, $a = x$

$$\frac{dy}{dx} = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+x^2}.$$

$$y = \sqrt{\frac{2}{\pi}} \int \frac{a}{a^2+x^2} dx \qquad \left[\because \int \frac{a}{a^2+x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$y = \sqrt{\frac{2}{\pi}} \cdot \cancel{a} \cdot \frac{1}{\cancel{a}} \tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ap}}{p} \sin px \, dp = \sqrt{\frac{2}{\pi}} \cdot \tan^{-1}\left(\frac{x}{a}\right) + c.$$

put $x = 0$.

$0 = 0 + c$

$\therefore c = 0$.

$$\therefore y = \sqrt{\frac{2}{\pi}} \cdot \tan^{-1}\left(\frac{x}{a}\right) + 0$$

$$y = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{x}{a}\right) = f(x) \quad \text{--- (1)}$$

(ii) Also deduce that $F_s^{-1}\left(\frac{1}{p}\right)$

put $a = 0$ in equation (1).

~~$f(x) = F_s^{-1}$~~

$$f(x) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{x}{0}\right)$$

e^0 in que

$$F_s^{-1}\left(\frac{1}{p}\right) = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2}.$$

$$= \sqrt{\frac{\pi}{2}}$$

Q: Find the fourier transform of $f(x) = \begin{cases} 1-|x| , & |x|<1 \\ 0 , & |x|>1 \end{cases}$

Hence deduce that $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x) . e^{ipx} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^1 (1-|x|) e^{ipx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)(\cos px + i\sin px)\,dx$$

$\hookrightarrow$ odd function $\to \int^{1} = 0.$

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \cdot 2\int_0^1 \underset{u}{(1-x)} \cdot \underset{v}{\cos px}\,dx.$$

$$= \frac{2}{\sqrt{2\pi}} \left[(1-x) \cdot \frac{\sin px}{p} - \int (0-1) \cdot \frac{\sin px}{p}\,dx \right]_0^1$$

$$= \frac{2}{\sqrt{2\pi}} \left[(1-1) - 0 + \left(\frac{-\cos px}{p^2} \right) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left\{ \frac{(1-\cos p)}{p^2} \right.$$

$$= F(P).$$

Deduction:

Using parseval's identity,

$$\int_{-\infty}^{\infty} |F(P)|^2\,dp = \int_{-\infty}^{\infty} |f(x)|^2\,dx.$$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{(1-\cos p)^2}{p^4}\,dp = \int_{-1}^{1} (1-|x|)^2\,dx$$

$$= 2\int_0^1 (1-x)^2\,dx.$$

$$= 2\int_0^1 (1-x)^2\,dx$$

$$= 2\left[\frac{(1-x)^3}{-3} \right]_0^1$$

$$= \frac{-2}{3}\left[(1-1)^3 - (1-0)^3 \right] = \frac{-2}{3}(0-1) \Rightarrow \frac{2}{3}$$

In LHS put $p = 2t$

$dp = 2dt$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{(1-\cos 2t)^2}{(2t)^4} \cdot 2dt = \frac{2}{3}$$

$$\frac{2}{\pi} \cdot 2\int_0^{\infty} \frac{(2\sin^2 t)^2}{16 \cdot t^4}\,dt = \frac{1}{3}$$

$$\frac{\cancel{16}}{\pi} \int_0^{\infty} \frac{\sin^4 t}{\cancel{16} t^4}\,dt = \frac{1}{3}$$

$$\therefore \int_0^{\infty} \frac{\sin^4 t}{16 \cdot t^4}\,dt = \frac{\pi}{3}$$

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Q: Show that the fourier transform of $e^{-x^2/2}$ is self-reciprocal.

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ipx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cdot e^{ipx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2} + ipx}\, dx. \rightarrow$$

$$-\frac{x^2}{2} + ipx$$

$$= -\frac{1}{2}[x^2 - 2ipx]$$

$$= -\frac{1}{2}[\underbrace{x^2 - 2ipx + i^2p^2} - i^2p^2]$$

$$= -\frac{1}{2}\left[(x-ip)^2 - \frac{p^2}{2}\right]$$

$$= -\frac{1}{2}(x-ip)^2 - \frac{p^2}{2}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-ip)^2}{2}} \cdot e^{-\frac{p^2}{2}} \cdot dx$$

$$= \frac{e^{-\frac{p^2}{2}}}{\sqrt{\cancel{2}\pi}} \int_{-\infty}^{\infty} e^{-y^2} \sqrt{\cancel{2}} \cdot dy$$

$$= \frac{e^{-\frac{p^2}{2}}}{\sqrt{\cancel{\pi}}} \cdot \sqrt{\cancel{\pi}}$$

$$= e^{-p^2/2}$$

$$= F(P),$$

put $\frac{(x-ip)^2}{2} = y^2$

$$\frac{x-ip}{\sqrt{2}} = y$$

Diff, $\frac{dx}{\sqrt{2}} = dy$

Q: Find the Inverse Fourier Transform of $F(P) = e^{-|p|y}$

From the inverse fourier transforms, we have.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} \cdot F(P)\, dp.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-(-Py)} \cdot e^{-ipx}\, dp + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-Py} \cdot e^{-ipx}\, dp.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{P(y-ix)}\, dp + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-P(y+ix)}\, dp.$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{P(y-ix)}}{y-ix}\right]_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-P(y+ix)}}{-(y+ix)}\right]_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{0} - e^{-\infty}}{y-ix}\right] + \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-\infty} - e^{0}}{-(y+ix)}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1-0}{y-ix} + \frac{0-1}{-(y+ix)}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{y-ix} + \frac{1}{y+ix}\right] = \frac{1}{\sqrt{2\pi}} \left[\frac{y + \cancel{ix} + y - \cancel{ix}}{y^2 - i^2x^2}\right] \quad [\because i^2 = -1]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2y}{(y^2+x^2)},$$

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Q:
Find the Fourier transform of $f(x) = \begin{cases} 0 & x < a \\ 1 & a < x < b \\ 0 & x > b. \end{cases}$

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{ipx} \, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{a}^{b} 1 \cdot e^{ipx} \, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{ipx}}{ip} \right]_{a}^{b}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{ipb} - e^{ipa}}{ip} \right]_{//}$$

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Z-transforms.

Def: Let $[f(n)]$ be a sequence defined for all +ve integers n. Then the z-transform of $f(n)$ is defined as

$$z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n} = F(z) \quad \text{--- (1)}$$

where z is an arbitrary complex no. This is one-sided z-transform.

Note: 1) If $f(n)$ is defined for $n = 0, \pm 1, \pm 2, \ldots$ then

$$z\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n) z^{-n}$$ and is called two sided transform.

2) If $f(n) = 0$ for $n < 0$, $\{f(n)\}$ is called a casual sequence.

3) The infinite series on RHS of (1) will be convergent only for certain values of z depending on the sequence $[f(n)]$. The inverse z-transform of $z[f(n)] = F(z)$ is defined as

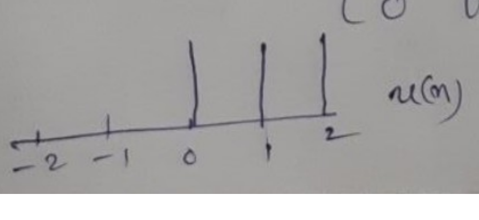
$$z^{-1}[f(z)] = f(n)$$

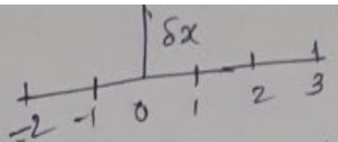
4) $\{\ \}$ is used for a sequence.

Unit sample sequence δx.

It is defined as the sequence with values $\delta x = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$

The unit step sequence $u(n)$ has values $u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0. \end{cases}$





Note: 1) In discrete systems, the input signal $f(t)$ is sampled at discrete instant $0, T, 2T, \ldots nT \ldots$ where T is sampling period. For such functions the z-transform (one sided) becomes

$$z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n} = F(z)$$

2) while finding z-transform

(i) If $f(n)$ is given, simply replace $f(n)$.

(ii) If $f(t)$ is given then replace t by nT.

---

1) $z[\delta(n)] = 1$

$$z[\delta(n)] = \sum_{n=0}^{\infty} \delta(n) z^{-n}$$

$$= 1 \cdot z^0 + 0 \cdot z^{-1} + \cdots$$

$$= 1.$$

---

2) $z\{u(n)\} = \frac{z}{z-1}$ for $|z| > 1$

$$\text{LHS} = \sum_{n=0}^{\infty} u(n) \cdot z^{-n} = \sum 1 \cdot z^{-n}.$$

$$= \sum \frac{1}{z^n} = \frac{1}{1 - \frac{1}{z}} \quad \text{if } |z^{-1}| < 1$$

$$= \frac{z}{z-1} \quad \text{if } |z| > 1$$

---

3) $z\{a^n f(t)\} = \sum_{n=0}^{\infty} a^n f(nT) z^{-n}$

$$= \sum_{n=0}^{\infty} f(nT) \left(\frac{z}{a}\right)^{-n}$$

$$= F\left(\frac{z}{a}\right).$$

3) $z\{a^n u(n)\} = \frac{z}{z-a}, \quad |z|>a$

$LHS = \sum_{n=0}^{\infty} a^n (1) z^{-n} = \sum \left(\frac{a}{z}\right)^n$

$= \frac{1}{1-\frac{a}{z}}$ if $\left|\frac{a}{z}\right|<1$

$= \frac{z}{z-a}$ if $|z|>a$

---

5) $z\{n f(t)\} = -z \cdot \frac{dF(z)}{dz}$

$z\{f(t)\} = \sum_{n=0}^{\infty} f(nT) z^{-n} = F(z)$

$\frac{dF(z)}{dz} = \sum_{n=0}^{\infty} -n f(nT) z^{-n-1}$

$z \cdot \frac{dF(z)}{dz} = -\sum_{n=0}^{\infty} n f(nT) z^{-n}$

$= -z(n f(t))$

---

6) $z\{f(n+p)T\} = z\{f(nT+pT)\}$

$= z^p \left[F(z) - f(a) - \frac{f(T)}{z} - \frac{f(2T)}{z^2} - \cdots - \frac{f\{(p-1)T\}}{z^{p-1}}\right]$

Pf: $LHS = z\{f(n+p)T\}$

$= \sum_{n=0}^{\infty} f((n+p)T) z^{-n}$

$= z^p \sum_{n=0}^{\infty} f\{(n+p)T\} z^{-(n+p)}$

$= z^p \sum_{m=p}^{\infty} f(mT) z^{-m}$

$= z^p \left[\sum_{m=p}^{\infty} f(mT) z^{-m}\right]$ where $m = n+p$.

$= z^p \left[\sum_{m=0}^{\infty} f(mT) z^{-m} - \sum_{m=0}^{p-1} f(mT) z^{-m}\right]$

$= z^p \left[F(z) - f(0) - \frac{f(T)}{z} - \frac{f(2T)}{z^2} - \cdots - \frac{f(p-1)T}{z^{p-1}}\right]$

Note: If $f(n+KT)$ is denoted by f_{n+k} then

$z[f(n+KT)] = z(f_{n+k})$

$= z^k\left[F(z) - f_0 - \frac{f_1}{z} - \frac{f_2}{z^2} - \cdots - \frac{f_{k-1}}{z^{k-1}}\right]$

---

1) $z(K) = \sum_{n=0}^{\infty} K z^{-n} = K \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$

$= K \cdot \frac{1}{1-\frac{1}{z}} = \frac{Kz}{z-1}$ if $|z|>1$

when $K=1$, $z(1) = \frac{z}{z-1}$

2) $z(-1)^n = \sum_{n=0}^{\infty} (-1)^n z^{-n} = \sum_{n=0}^{\infty} (-z)^{-n}$

$= \frac{1}{1+\frac{1}{z}} = \frac{z}{z+1}$ if $|z|>1$

3) $z(a^n) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum \left(\frac{a}{z}\right)^n$

$= \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}$ if $|z|>|a|$

4) $z\{n\} = \sum_{n=0}^{\infty} n z^{-n}$

$= -z \cdot \frac{d}{dz}[z(1)] = -z \cdot \frac{d}{dz}\left(\frac{z}{z-1}\right)$

$= \frac{z}{(z-1)^2}$

Assig. 4

6) To find $z(\cos n\theta)$, $z(\sin n\theta)$

we know that

$$z(a^n) = \frac{z}{z-a} \quad \text{if } |z|>|a|$$

put $a = e^{i\theta}$

$$z(a^n) = z(e^{in\theta}) = \frac{z}{z-e^{i\theta}}$$

$$= \frac{z}{z-(\cos\theta + i\sin\theta)}$$

$$= \frac{(z^2 - z\cos\theta) + iz\sin\theta}{(z-\cos\theta)^2 + \sin^2\theta}$$

$$= \frac{z^2 - z\cos\theta + iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Equating the real and imaginary we get.

7) To find $z(r^n \cos n\theta)$ and $z(r^n \sin n\theta)$.

Sol: $z(a^n) = \frac{z}{z-a}$ if $|z|>|a|$

put $a = re^{i\theta}$

$$z(a^n) = z(r^n e^{in\theta}) = \frac{z}{z-re^{i\theta}}$$

$$= \frac{z(z-r\cos\theta) + ri\sin\theta}{(z-r\cos\theta)^2 - (ri\sin\theta)^2}$$

$$= \frac{z(z-r\cos\theta) + zri\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

Equating real and imagin.

8) $$z(n^2) = -z\frac{d}{dz}(z(n))$$

$$= -z\cdot\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right)$$

$$= \frac{z(z+1)}{(z-1)^3}$$

Standard discrete fns.

1) $$z(t) = \sum_{n=0}^{\infty}(nT)z^{-n}$$

$$= T\sum_{n=0}^{\infty} nz^{-n}$$

$$= T\left[\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots\right]$$

$$= \frac{T}{z}\cdot\left(1-\frac{1}{z}\right)^{-2} = \frac{Tz}{(z-1)^2}$$

2) $$z(e^{-at}) = \sum_{n=0}^{\infty}\left(e^{-anT} z^{-n}\right)$$

$$= \sum_{n=0}^{\infty}\left(e^{-aT}\right)^n z^{-n}$$

$$= \sum\left(\frac{e^{-aT}}{z}\right)^n$$

$$= 1 + \frac{e^{-aT}}{z} + \cdots \qquad \left|\frac{e^{-aT}}{z}\right| < 1$$

$$= \frac{1}{1-\frac{e^{-aT}}{z}}$$

$$= \frac{z}{z-e^{-aT}}$$

3) $$z(\cos(\omega t)) = \sum_{n=0}^{\infty}[\cos(n\omega T)]z^{-n}$$

$$= z(\cos n\theta) \quad \text{when } \theta = \omega T$$

$$= \frac{z(z-\cos\omega T)}{z^2 - 2z\cos\omega T + 1}$$

4) $$z[\sin(\omega t)] = \sum \sin(n\omega T)z^{-n}$$

$$= z(\sin n\theta) \quad \text{when } \theta = \omega T$$

$$= \frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1}$$

5) $z(t^k) = \sum (nT)^k \cdot z^{-n}$ ⊗

$= Tz \sum n^k \cdot T^{k-1} \cdot z^{-(n+1)}$

$= Tz \sum (nT)^{k-1} \cdot n\, z^{-(n+1)}$ —(1)

continued

First shifting theorem

If $z\{f(t)\} = F(z)$ then

$z[e^{-at} \cdot f(t)] = F(ze^{aT})$

Sol

$LHS = \sum e^{-anT} \cdot f(nT)\, z^{-n}$

$= \sum f(nT)\,(ze^{aT})^{-n}$

$= F(ze^{aT})$

Second shifting theorem:

If $z\{f(t+T)\} = z[F(z) - f(0)]$

Pf: $z\{f(t+T)\}$

$= \sum f(nT+T)\, z^{-n}$

$= z \sum_{n=0}^{\infty} f((n+1)T)\, z^{-(n+1)}$

$= z \sum_{k=1}^{\infty} f(kT)\, z^{-k}$

(by taking $k = n+1$)

$= z\left[\sum_{k=0}^{\infty} f(kT)\, z^{-k} - f(0)\right]$

$= z[z(f(t)) - f(0)]$

$= z[F(z) - f(0)]$

Similarly

$z(t^{k-1}) = \sum (nT)^{k-1} \cdot z^{-n}$

$\frac{d}{dz}[z(t^{k-1})]$

$= \sum (nT)^{k-1} \cdot (-n)\, z^{-(n+1)}$

$= -\sum (nT)^{k-1} \cdot n\, z^{-(n+1)}$ —(2)

From (1) and (2),

$z(t^k) = -Tz \cdot \frac{d}{dz}[z(t^{k-1})]$

put $k=1$

$z(t) = -Tz \cdot \frac{d}{dz}[z(1)]$

$= -Tz \cdot \frac{d}{dz}\left(\frac{z}{z-1}\right)$

$= \frac{Tz}{(z-1)^2}$

Put $k=2$,

$z(t^2) = -Tz \cdot \frac{d}{dz}[z(t)]$

$= -Tz \cdot \frac{d}{dz}\left[\frac{Tz}{(z-1)^2}\right]$

$= \frac{T^2 z\,(z+1)}{(z-1)^3}$

Initial value theorem:

If $z[f(t)] = F(z)$ then

$f(0) = \lim_{z\to\infty} F(z)$

Pf: $F(z) = z(f(t))$

$= \sum_{n=0}^{\infty} f(nT)\, z^{-n}$

$= f(0) + \frac{f(T)}{z} + \frac{f(2T)}{z^2} + \cdots$

Taking limit as $z \to \infty$,

$\lim_{z\to\infty} F(z) = f(0)$

Final value theorem:

If $z(f(t)) = F(z)$ then

$$\lim_{t\to\infty} f(t) = \lim_{z\to 1} (z-1) F(z)$$

Pf: $z[f(t+T) - f(t)]$

$$= \sum_{n=0}^{\infty} [f(n+1)T - f(nT)] z^{-n}$$

$$zF(z) - zf(0) - F(z)$$

$$= \sum_{n=0}^{\infty} [f(n+1)T - f(nT)] z^{-n}$$

$$\lim_{z\to 1} [(z-1)F(z) - zf(0)]$$

$$= \lim_{z\to 1} \left[[f(n+1)T - f(nT)] z^{-n} \right]$$

$$\lim_{z\to 1} (z-1) f(z) - 1.f(0)$$

$$= \lim_{z\to 1} \left[\sum_{n=0}^{\infty} [f(n+1)T - f(nT)] \right]$$

$$= \lim_{n\to\infty} [f(T) - f(0) + f(2T) - f(T) + \dots + f((n+1)T) - f(nT)]$$

$$= \lim_{n\to\infty} [f(n+1)T - f(0)]$$

$$= f(\infty) - f(0)$$

$$= \lim_{t\to\infty} f(t) - f(0)$$

$$\therefore \lim_{z\to 1} (z-1) F(z) = \lim_{t\to\infty} f(t).$$

h) $Z\{\delta(n-k)\}$.

$$= \sum_{n=0}^{\infty} \delta(n-k) . z^{-n}$$

$$= \frac{1}{z^k}.$$

$$\delta(n-k) = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n. \end{cases}$$

(i) $Z\left\{\left(\frac{1}{4}\right)^n u(n)\right\} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n u(n) . z^{-n}$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n z^{-n} = \sum_{n=0}^{\infty} (4z)^{-n}.$$

$$= 1 + \frac{1}{4z} + \frac{1}{(4z)^2} + \dots$$

$$= \frac{1}{1 - \frac{1}{4z}} = \frac{4z}{4z-1}.$$

(j) $z(f(n))$ if $f(n) = \begin{cases} 0 & \text{if } n > 0 \\ 1 & \text{if } n \le 0. \end{cases}$

$$z(f(n)) = \sum_{n=-\infty}^{\infty} f(n) . z^{-n}$$

$$= \sum_{0}^{\infty} z^n = \frac{1}{1-z} \quad \text{if } |z| < 1.$$

(k) $ab^n \quad (a \neq 0, \ b \neq 0)$

$$z(ab^n) = \sum_{n=0}^{\infty} ab^n . z^{-n} = a . \sum_{n=0}^{\infty} \left(\frac{b}{z}\right)^n$$

$$= a\left(\frac{z}{z-b}\right) \quad \text{if } |z| > b.$$

Using shifting theorem or otherwise, find the z-transform

(a) $z(e^{-at}) = z(e^{-at}\cdot 1)$

$= \left[Z(1)\right]_{z \to ze^{aT}}$

$= \left(\frac{z}{z-1}\right)_{z \to ze^{aT}}$

$= \frac{ze^{aT}}{ze^{aT}-1} = \frac{z}{z-e^{-aT}}$

(b) $Z(te^{-at}) = \left[Z(t)\right]_{z \to ze^{aT}}$

$= \left[\frac{Tz}{(z-1)^2}\right]_{z \to ze^{aT}}$

$= \frac{Tze^{aT}}{(ze^{aT}-1)^2} = \frac{Tze^{-aT}}{(z-e^{-aT})^2}$

(c) $\overset{\cos at}{Z}(e^{-iat}) = \frac{z}{z-e^{-iaT}} \times \frac{(z-e^{iaT})}{(z-e^{iaT})}$

$= \frac{z(z-\cos aT + i\sin aT)}{z^2 - 2z\cos aT + 1}$

$z(\cos at) = \frac{z(z-\cos aT)}{z^2 - 2z\cos aT + 1}$

$z(\sin at) = \frac{z\sin aT}{z^2 - 2z\cos aT + 1}$

(d) $z(e^{-at}\cos bt) = \left[Z(\cos bt)\right]_{z \to ze^{aT}}$

$= \left[\frac{z(z-\cos bT)}{z^2 - 2z\cos bT + 1}\right]_{z \to ze^{aT}}$

$= \frac{ze^{aT}(ze^{aT} - \cos bT)}{z^2e^{2aT} - 2ze^{aT}\cos bT + 1}$

$z(e^{-at}\sin bt) = \frac{ze^{aT}\sin bT}{z^2e^{2aT} - 2ze^{aT}\cos bT + 1}$

Find the z-transform of

1) t^2e^{-t}.

$z(t^2e^{-t}) = \left[Z(t^2)\right]_{z \to ze^{T}}$

$= \left[\frac{T^2z(z+1)}{(z-1)^3}\right]_{z \to ze^{T}}$

$= \frac{T^2(ze^T)(ze^T+1)}{(ze^T-1)^3}$

2) $Z(e^{at+b}) = Z(e^{at}\cdot e^{b})$

$= e^b \cdot Z(e^{at}) = e^b\cdot\left[Z(1)\right]_{z \to ze^{-aT}}$

$= e^b\left(\frac{z}{z-1}\right)_{z \to ze^{-aT}}$

$= e^b\cdot\left(\frac{ze^{-aT}}{ze^{-aT}-1}\right)$ if $|z| > e^{aT}$

$= e^b\cdot\left(\frac{z}{z-e^{aT}}\right)$

3) $z(e^{2t}\cos t) = Z(\cos t)_{z \to ze^{-2T}}$

$= \left[\frac{z(z-\cos T)}{z^2 - 2z\cos T + 1}\right]_{z \to ze^{-2T}}$

$= \frac{ze^{-2T}(ze^{-2T} - \cos T)}{z^2e^{-4T} - 2ze^{-2T}\cos T + 1}$

1) $Z(e^{-2t}t^3) = \left[Z(t^3)\right]_{z \to ze^{2T}}$

$$= \left[\frac{T^3(z^3+4z^2+z)}{(z-1)^4}\right]_{z \to ze^{2T}}$$

$$= \frac{T^3\left[z^3e^{6T} + 4ze^{4T} + ze^{2T}\right]}{(ze^{2T}-1)^4}$$

---

2) $Z\left[a^n \cos n\pi\right] = \sum_{n=0}^{\infty} a^n (\cos n\pi) z^{-n}$

$$= 1 - \frac{a}{z} + \left(\frac{a}{z}\right)^2 - \left(\frac{a}{z}\right)^3 + \cdots$$

$$= \frac{1}{1+\frac{a}{z}} \quad \text{if } \left|\frac{a}{z}\right| < 1$$

$$= \frac{z}{z+a} \quad \text{if } |z| > |a|$$

---

3) Find the z-transform of

$$f(n) = \begin{cases} \frac{a^n}{n!} & \text{for } n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{n!} \cdot z^{-n}$$

$$= 1 + \frac{\left(\frac{a}{z}\right)}{1!} + \frac{\left(\frac{a}{z}\right)^2}{2!} + \cdots = e^{a/z}$$

$$Z(n^2a^n) = \frac{az^2+a^2z}{(z-a)^3}$$

$$Z\left(\frac{1}{n}\right) = \log\left(\frac{z}{z-1}\right)$$

$$Z(r^n \cos n\theta) = \frac{z(z - r\cos\theta)}{z^2 - 2zr\cos\theta + r^2}$$

$$Z(r^n \sin n\theta) = \frac{rz\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

$$Z\left(a^n \cos\frac{n\pi}{2}\right) = \frac{z^2}{z^2+a^2}$$

$$Z\left(a^n \sin\frac{n\pi}{2}\right) = \frac{az}{z^2+a^2}$$

$f(n)$	$Z[f(n)]$
1	$\frac{z}{z-1}$
$(-1)^n$	$\frac{z}{z+1}$
$u(n)$	$\frac{z}{z-1}$ if $\lvert z\rvert>1$
$\delta(n)$	1
$a^n u(n)$ or a^n	$\frac{z}{z-a}$ if $\lvert z\rvert > a$
K	$\frac{Kz}{z-1}$
n	$\frac{z}{(z-1)^2}$
n^2	$\frac{z^2+z}{(z-1)^3}$
$n(n-1)$	$\frac{2z}{(z-1)^3}$
n^3	$\frac{z^3+4z^2+z}{(z-1)^4}$
na^n	$\frac{az}{(z-a)^2}$
$\cos n\theta$ or $u(n)\cos n\theta$	$\frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}$
$\sin n\theta$ or $u(n)\sin n\theta$	$\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$
$\frac{a^n}{n!}$	$e^{a/z}$
t	$\frac{Tz}{(z-1)^2}$
t^2	$\frac{T^2z(z+1)}{(z-1)^3}$
t^3	$\frac{T^3(z^3+4z^2+z)}{(z-1)^4}$
e^{-at}	$\frac{z}{z-e^{-aT}}$ if $\lvert z\rvert > e^{-aT}$

Inverse z-transform

Since z-transform is defined as $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} = F(z)$, (one-sided) to find the inverse z-transform $f(n) = z^{-1}\{F(z)\}$, expand $F(z)$ in the proper power series and collect the coefficient of z^{-n} to get $f(n)$

Method 1 : (Long division method)

a) Find the inverse z-transform of $\frac{z}{(z-2)(z-3)} = \frac{z}{z^2-5z+6}$

$$= \frac{z^{-1}+5z^{-2}+19z^{-3}}{1-5z^{-1}+6z^{-2}}$$

$$\begin{array}{r|l} 1-5z^{-1}+6z^{-2} & z^{-1} \\ & \underline{z^{-1}} \\ & z^{-1}-5z^{-2}+6z^{-3} \\ & \overline{5z^{-2}-6z^{-3}} \\ & 5z^{-2}-25z^{-3}+30z^{-4} \\ & \overline{19z^{-3}-30z^{-4}} \\ & 19z^{-3}-95z^{-4}+114z^{-5} \\ & \overline{-125z^{-4}-114z^{-5}} \end{array}$$

$F(z) = z^{-1}+5z^{-2}+19z^{-3}+\dots$

comparing the quotient with

$$\sum_{n=0}^{\infty} f(n) z^{-n} = f(0) + z^{-1}f(1) + z^{-2}f(2)$$

we have $f(0)=0$, $f(1)=1$, $f(2)=5$

$f(3)=19$

$\therefore f(n) = 3^n - 2^n$

2) $\frac{2z(z^2-1)}{(z^2+1)^2} = \frac{2z^{-1}-2z^{-3}}{1+2z^{-2}+z^{-4}}$

By actual division

$F(z) = 2z^{-1} - 6z^{-3} + 10z^{-5} - 14z^{-7} \dots$

comparing coeff with

$$\sum_{n=0}^{\infty} f(n) z^{-n} = f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \dots$$

$f(0)=0$ $f(1)=2$ $f(2)=0$

$f(6)=-6$ $f(4)=0$ $f(5)=10$

$f(6)=0$ $f(2)=-14$

$f(n) = 2n\sin\left(\frac{n\pi}{2}\right)$

(It is indeed difficult to give the)

Find the inverse z transform of

$$\frac{z}{z^2+9z+20} = \frac{z^{-1}}{1+9z^{-1}+20z^{-2}}$$

By actual division,

$f(0)=0$, $f(1)=1$, $f(2)=-9$, $f(3)=-61$

Method-2

Partial fraction method.

$$F(z) = \frac{z}{z^2+9z+20}$$

$$\frac{F(z)}{z} = \frac{1}{(z+4)(z+5)}$$

$$F(z) = \frac{z}{z+4} - \frac{z}{z+5}$$

$$f(n) = z^{-1}(F(z))$$

$$= z^{-1}\left(\frac{z}{z+4} - \frac{z}{z+5}\right)$$

$= (-4)^n - (-5)^n$ since

$$z(a^n) = \frac{z}{z-a}.$$

2) $F(z) = \frac{z}{(z+2)(z-1)^2}$

$= \frac{1}{9}\left(\frac{z}{z+2}\right) - \frac{1}{9}\left(\frac{z}{z-1}\right) + \frac{1}{3}\left(\frac{z}{(z-1)^2}\right)$

$f(n) = Z^{-1}(F(z))$

$= \frac{1}{9}(-2)^n - \frac{1}{9}(1)^n + \frac{1}{3}n \quad \because Z(n) = \frac{z}{(z-1)^2}$

---

3) $F(z) = \frac{6z^2}{(2z-1)(3z+2)}$

$\frac{F(z)}{z} = \frac{z}{\left(z-\frac{1}{2}\right)\left(z+\frac{2}{3}\right)}$

$= \frac{3}{7}\left(\frac{z}{z-\frac{1}{2}}\right) + \frac{4}{7}\left(\frac{z}{z+\frac{2}{3}}\right)$

$f(n) = z^{-1}(F(z))$

$= \frac{3}{7}\left(\frac{1}{2}\right)^n + \frac{4}{7}\left(-\frac{2}{3}\right)^n.$

1) multiplication by n.

If $Z[u_n] = \bar{u}(z)$ then

$Z[n u_n] = -z \cdot \frac{d}{dz}[\bar{u}(z)]$

---

2) Division by n

$Z\left[\frac{u_n}{n}\right] = -\int \frac{\bar{u}(z)}{z} dz.$

---

3) Initial value theorem.

$\lim_{z\to\infty} \bar{u}(z) = u_0.$

similarly u_1, u_2, u_3 can be calculated.

$u_2 = \lim_{z\to\infty} z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right]$

$u_3 = \lim_{z\to\infty} z^3\left[\bar{u}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2}\right]$

Def: The z-transform of a sequence $\{u_n\}$ is defined by $Z[\{u_n\}] = \sum_{n=0}^{\infty} \frac{u_n}{z^n} = \bar{u}(z)$ say.

1) Damping rule.

$Z[a^n u_n] = \bar{u}\left(\frac{z}{a}\right)$

Ex: 1) $Z\left[\frac{1}{n!}\right] = e^{1/z}.$

$Z\left[\frac{a^n}{n!}\right] = e^{a/z}.$

2) $Z[a^n] = \frac{z}{z-a}.$

$Z[e^{in\pi/2}] = \frac{z}{z - e^{i\pi/2}}$

$= \frac{z}{z-i} = \frac{z(z+i)}{z^2+1}.$

$\therefore Z\left[\cos\frac{n\pi}{2}\right] = \frac{z^2}{z^2+1}$

$Z\left[\sin\frac{n\pi}{2}\right] = \frac{z}{z^2+1}$

$\bar{u}(z) = u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \cdots$

$\lim_{z\to\infty} \bar{u}(z) = u_0.$

$z \cdot \bar{u}(z) = z u_0 + u_1 + \frac{u_2}{z} + \cdots$

$\lim_{z\to\infty} z[\bar{u}(z) - u_0] = u_1$

Ass 4/2 If $z(n) = \frac{z}{(z-1)^2} = \bar{u}(z)$

$$z(n^2) = -z\cdot\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= \frac{z^2+z}{(z-1)^3}$$

2) find $z(n+2)$

$$Z\left[u_{n+k}\right] = z^k\left[\bar{u}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} \cdots - \frac{u_{k-1}}{z^{k-1}}\right]$$

Here $u_n = n$.

For $k=2$

$$z[n+2] = z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right]$$

$u_0=0$, $u_1=1$, $\bar{u}(z) \overset{= z(n)}{=} \frac{z}{(z-1)^2}$

$$z[n+2] = z^2\left[\frac{z}{(z-1)^2} - 0 - \frac{1}{z}\right]$$

$$= \frac{z(2z-1)}{(z-1)^2}$$

Alternate method.

$$z[n+2] = z[n] + z[2]$$

$$= \frac{z}{(z-1)^2} + \frac{2z}{(z-1)}$$

3) If $z[2^n] = \frac{z}{z-2}$ $\left(\frac{2}{z}<1\right)$

find $z[n.2^n]$

If $z[u_n] = \bar{u}(z)$ then

$$z[n.u_n] = -z\cdot\frac{d}{dz}(\bar{u}(z))$$

$$= \frac{2z}{(z-2)^2}$$

4) If $z[u_n] = \frac{z}{z-1} + \frac{z}{z^2+1} = \bar{u}(z)$

find $z[u_{n+2}]$

$$u_0 = \lim_{z\to\infty}\bar{u}(z) = 1+0 = 1$$

$$u_1 = \lim_{z\to\infty} z\left[\bar{u}(z) - u_0\right]$$

$$= 2$$

$$z[u_{n+2}] = z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right]$$

$$= z^2\left[\frac{z}{z-1} + \frac{z}{z^2+1} - 1 - \frac{2}{z}\right]$$

$$= \frac{z(z^2-z+2)}{(z-1)(z^2+1)}$$

5) If $z[n^2] = \frac{z(z+1)}{(z-1)^3} = \bar{u}(z)$

find $z(n^3)$

$$= -z\cdot\frac{d}{dz}(\bar{u}(z))$$

$$= \frac{z(z^2+4z+1)}{(z-1)^4}.$$

6) If $z[a^n] = \frac{z}{z-a}$. find $= \bar{u}(z)$

$$z[n.a^n] = -z\cdot\frac{d}{dz}(\bar{u}(z))$$

$$= -z\cdot\frac{d}{dz}\left(\frac{z}{z-a}\right) = \frac{az}{(z-a)^2}$$

Ass 4/3 7) If $z[u_n] = \frac{2z^2+3z+4}{(z-3)^3}$

find u_0, u_1, u_2, u_3.

Sol. If $z[u_n] = \bar{u}(z)$ then

$$u_0 = \lim_{z\to\infty}\bar{u}(z) = 0$$

$$u_1 = \lim_{z\to\infty} z\left[\bar{u}(z) - u_0\right] = 2$$

$$u_2 = \lim_{z\to\infty} z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right]$$

$$= \lim_{z\to\infty} z^2\left[\frac{2z^2+3z+4}{(z-3)^3} - \frac{2}{z}\right]$$

$$= \lim_{z\to\infty} \frac{z\left[21z^2-50z+54\right]}{(z-3)^3} = 21$$

$$u_3 = \lim_{z\to\infty} z^3\left[\bar{u}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{2z^2+3z+4}{(z-3)^3} - \frac{2}{z} - \frac{21}{z^2}\right]$$

$$= 139$$

8) If $z[u_n] = \dfrac{(z^2+4z+1)z}{(z-1)^4}$

find $z\left[\dfrac{u_n}{n}\right]$

Soln: If $z[u_n] = \bar{u}(z)$ then

$$z\left[\frac{u_n}{n}\right] = -\int \frac{\bar{u}(z)}{z}\,dz$$

$$= -\int \frac{z^2+4z+1}{(z-1)^4}\,dz$$

$$= -\int\left[\frac{1}{(z-1)^2} + \frac{6}{(z-1)^3} + \frac{6}{(z-1)^4}\right]dz$$

$$= \frac{z^2+z}{(z-1)^3}$$

9) If $z[u_n] = \dfrac{z^2+2z+6}{(z-1)^3} = \bar{u}(z)$

Find u_0, u_1, u_2

Soln: $u_0 = \lim\limits_{z\to\infty} \bar{u}(z) = \lim\limits_{z\to\infty} \dfrac{z^2 \, 2z+6}{(z-1)^3} = 0$

$$u_1 = \lim_{z\to\infty} z\left[\bar{u}(z) - u_0\right]$$

$$= \lim_{z\to\infty} \frac{z(z^2+2z+6)}{(z-1)^3} = 1$$

$$u_2 = \lim_{z\to\infty} z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right]$$

$$= \lim_{z\to\infty} z^2\left[\frac{z^2+2z+6}{(z-1)^3} - \frac{1}{z}\right]$$

$$= \lim_{z\to\infty} \frac{z\left[5z^2+3z+1\right]}{(z-1)^3} = 5$$

Convolution theorem.

If $z^{-1}[\bar{u}(z)] = u_n$

$z^{-1}[\bar{V}(z)] = V_n$ then.

$$z^{-1}\left[\bar{u}(z).\,\bar{V}(z)\right] = \sum_{m=0}^{\infty} u_m V_{n-m}$$

$$= u_n * V_n$$

1) $z^{-1}\left[\dfrac{z^2}{(z-1)(z-3)}\right]$

$z^{-1}\left[\dfrac{z}{z-1}\right] = 1^n, \quad z^{-1}\left[\dfrac{z}{z-3}\right] = 3^n$

$$z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = 1^n * 3^n$$

by convolution theorem

$$= \sum_{m=0}^{n} 1^m . 3^{n-m}$$

$$= 3^n + 3^{n-1} + \dots + 3 + 1$$

$$= \frac{(3^{n+1}-1)}{2}$$

2) $\dfrac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$

$$(z) = \frac{2z \,.\, 4z}{(2z-1)(4z-1)}$$

$$= \frac{\cancel{2}z}{\cancel{2}\left(z-\frac{1}{2}\right)} \cdot \frac{\cancel{4}z}{\cancel{4}\left(z-\frac{1}{4}\right)}$$

Let $F(z) = \frac{z}{z-\frac{1}{2}}$

$f_n = z^{-1}\{F(z)\} = \left(\frac{1}{2}\right)^n.$

$G(z) = \frac{z}{z-\frac{1}{4}}$

$g_n = z^{-1}\{G(z)\} = \left(\frac{1}{4}\right)^n.$

From convolution theorem of inverse z-transform

$z^{-1}\{F(z).G(z)\} = f_n * g_n$

$= \left(\frac{1}{2}\right)^n * \left(\frac{1}{4}\right)^n$

$= \sum_{K=0}^{n} \left(\frac{1}{2}\right)^k \left(\frac{1}{4}\right)^{n-k}$

$= \sum 2^{-k} . 4^{-n+k}$

$= \sum 2^{-k} . 4^k . 4^{-n}.$

$= 4^{-n} \sum 2^k$

$= 2^{-2n}(2^0 + 2^1 + \dots + 2^n)$

$= 2^{-2n}\left(\frac{2^{n+1}-1}{2-1}\right) = 2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n$

---

1) Let f_n and g_n be two sequences such that $z[f_n] = F(z)$ and $z[g_n] = G(z)$. Then

$z[f_n * g_n] = F(z)\, G(z)$

2) convolution theorem of inverse z-transform.

If $z^{-1}[F(z)] = f(n)$ and $z^{-1}[G(z)] = g(n)$ then

$z^{-1}\{F(z). G(z)\} = f(n) * g(n)$

$= \sum_{K=0}^{n} f(k)\, g(n-k)$

2) $z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$

Let $F(z) = \frac{z}{z-a}$ $\quad G(z) = \frac{z}{z-b}$

$f_n = z^{-1}[F(z)]$ $\quad g_n = b^n$

$= a^n$

From convolution theorem of inverse z-transform

$z^{-1}\{F(z). G(z)\} = f_n * g_n$

$= a^n * b^n$

$= \sum_{K=0}^{n} a^k . b^{n-k} = b^n \sum_{K=0}^{n} \left(\frac{a}{b}\right)^k$

$= b^n\left[\frac{\left(\frac{a}{b}\right)^{n+1} - 1}{\frac{a}{b}-1}\right] = \frac{a^{n+1} - b^{n+1}}{a-b}$

Also Deduce. $z^{-1}\left[\frac{z^2}{(z-4)(z-5)}\right]$

$= \frac{4^{n+1} - 5^{n+1}}{-1} = 5^{n+1} - 4^{n+1}$

Translation theorem (or) Shifting rule

(i) If $z(u_n) = F(z)$ then.

$z(u_{n-k}) = z^{-k}. F(z)$

$z(u_{n+k}) = z^k\left[F(z) - \sum_{n=0}^{k-1} u_n z^{-n}\right]$

Find $z(2.3^n + 5.n)$

$= 2. \frac{1}{z-3} + 5. \frac{z}{(z-1)^2}$

$\because z(a^n) = \frac{1}{z-a}, \quad z(n) = \frac{z}{(z-1)^2}$

Applying shifting theorem

1) If $z[u_n] = \frac{(z^2+4z+1)z}{(z-1)^4}$

find

$$z\left[\frac{u_n}{n}\right] = -\int \frac{\bar{u}(z)}{z}\,dz$$

$$= -\int \frac{z^2+4z+1}{(z-1)^4}\,dz$$

$$= -\int\left[\frac{1}{(z-1)^2} + \frac{6}{(z-1)^3} + \frac{6}{(z-1)^4}\right]dz$$

$$= \frac{1}{z-1} + \frac{3}{(z-1)^2} + \frac{2}{(z-1)^3} = \frac{z^2+z}{(z-1)^3}$$

---

2) If $z[u_n] = \frac{z}{z-1} + \frac{z}{z^2+1}$

find $z[u_{n+2}]$

$$u_0 = \underset{z\to\infty}{Lt}\ \bar{u}(z) = 1 + 0 = 1.$$

$$u_1 = \underset{z\to\infty}{Lt}\ z[\bar{u}(z) - u_0]$$

$$= \underset{z\to\infty}{Lt}\ z\left[\frac{z}{z-1} + \frac{z}{z^2+1} - 1\right]$$

$$= \underset{z\to\infty}{Lt}\ \frac{z(2z^2-z^2)}{(z-1)(z^2+1)} = 2$$

$$z[u_{n+2}] = z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right]$$

$$= z^2\left[\frac{z}{z-1} + \frac{z}{z^2+1} - 1 - \frac{2}{z}\right]$$

$$= \frac{z(z^2-z+2)}{(z-1)(z^2+1)}$$

---

3) Find $z^{-1}\left[\frac{1}{z-2}\right]$ when $|z|>2$

$$\left|\frac{z}{2}\right| > 1 \Rightarrow \left|\frac{2}{z}\right| < 1$$

$$z[u_n] = \frac{1}{z-2} = \frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$= \frac{1}{z}\left[1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \cdots\right]$$

$$= \frac{1}{z} + \frac{2}{z^2} + \frac{2^2}{z^3} + \frac{2^3}{z^4} + \cdots$$

$u_1 = 2^0,\ u_2 = 2^1 \ldots$

$$u_n = 2^{n-1},\quad n\geq 1$$

$$z^{-1}\left[\frac{1}{z-2}\right] = 2^{n-1}\ (n\geq 1,\ |z|>2)$$

Find $z^{-1}\left[\frac{1}{z-2}\right]$ when $\left|\frac{z}{2}\right| < 1$

$$z[u_n] = \frac{1}{z-2}$$

$$= \frac{1}{-2\left(1-\frac{z}{2}\right)}$$

$$= -\frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \cdots\right.$$

Applications of z-transforms.

we can find the sequence $\{u_n\}$ if the recurrence relation among u_n, u_{n+1}, u_{n+2} which is called the difference equation is given. we can solve the difference eqn.

1) Solve the diff. eqn $u_{n+2} - 4u_{n+1} + 4u_n = 0$ (1) given that $u_0 = 1,\ u_1 = 0$.

Sol. $z[u_n] = \bar{u}(z)$

$$z[u_{n+1}] = z[\bar{u}(z) - u_0]$$

$$z[u_{n+2}] = z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right]$$

$\therefore$ (1) becomes.

$$z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] - 4z[\bar{u}(z) - u_0] + 4\bar{u}(z) = 0.$$

$$\bar{u}(z)[z^2 - 4z + 4] = z^2 - 4z$$

since $u_0 = 1,\ u_1 = 0$

$$\bar{u}(z) = \frac{z(z-4)}{(z-2)^2} = \frac{z}{z-2} - \frac{2z}{(z-2)^2}$$

$$\therefore u_n = 2^n - n.2^n$$

2) Solve the difference eqn

Assy.

$u_{n+2} + 4u_{n+1} + 3u_n = 3^n$

given that $u_0 = 0,\ u_1 = 1$.

Sol:

$z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right]$

$+ 4z\left[\bar{u}(z) - u_0\right] + 3\bar{u}(z) = \frac{z}{z-3}$

since $z(3^n) = \frac{z}{z-3}$.

$(z^2 - 4z + 3)\bar{u}(z) = z + \frac{z}{z-3}$

$\bar{u}(z) = \frac{z(z-2)}{(z-3)(z+1)(z+3)}$

$= \frac{1}{24}\cdot\frac{z}{z-3} + \frac{3}{8}\cdot\frac{z}{z+1} - \frac{5}{12}\cdot\frac{z}{(z+3)}$

$u_n = z^{-1}\{\bar{u}(z)\}$

$= \frac{1}{24}\cdot 3^n + \frac{3}{8}\cdot(-1)^n - \frac{5}{12}(-3)^n$.

Assy. 4

3) Solve the diff. eqn

$u_{n+2} - 2u_{n+1} + u_n = 2^n$ — (1).

given that $u_0 = 1,\ u_1 = 0$.

Soln: Taking z-transforms both sides

$z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] - 2z\left[\bar{u}(z) - u_0\right]$

$+ \bar{u}(z) = \frac{z}{z-2}$

$(z^2 - 2z + 1)\bar{u}(z) = \frac{z}{z-2} + z^2 - 2z$

$\bar{u}(z) = \frac{z(z^2 - 2z + 4)}{(z-2)(z-1)^2}$

$u_n = z^{-1}\{\bar{u}(z)\}$

$= z^{-1}\left\{\frac{4z}{z-2} - \frac{3z}{z-1} - \frac{3z}{(z-1)^2}\right\}$

$= 4(2)^n - 3(1)^n - 3.n.1^n$

$= 4\cdot 2^n - 3(1+n)$

4) Solve the D.E.

$6u_{n+2} - u_{n+1} - u_n = 0$ — (1)

$u(0) = 0 \quad u(1) = 1$

Soln: Taking z-transform both sides

$6z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] - z\left[\bar{u}(z) - u_0\right]$

$- \bar{u}(z) = 0$

$(6z^2 - z - 1)\bar{u}(z) = 6z$.

$\bar{u}(z) = \frac{6z}{5(z-\frac{1}{2})} - \frac{6z}{5(z+\frac{1}{3})}$

$u_n = \frac{6}{5}\left(\frac{1}{2}\right)^n - \frac{6}{5}\left(-\frac{1}{3}\right)^n$.

5) $4u_n - u_{n+2} = 0$ (1), $u_0 = 0,\ u_1 = 2$

$-z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] + 4\bar{u}(z) = 0$.

$\left[-z^2 + 4\right]\bar{u}(z) = -2z$

$\bar{u}(z) = \frac{-2z}{4 - z^2}$

$= \frac{2z}{4(z-2)} - \frac{2z}{4(z+2)}$

$u_n = \frac{1}{2}(2^n) - \frac{1}{2}(-2)^n$.

Method 3 (Residue method)

By using theory of complex no.s it can be shown that the inverse z transform is given by

$f(n) = \frac{1}{2\pi i}\int_C F(z)\cdot z^{n-1}\,dz$

where C is the circle (or some closed contour) which contains all the isolated singularities of $F(z)$ and containing the origin of the z-plane in the region of convergence. Hence by Cauchy's residue theorem

$f(n) =$ sum of the residues of the singularities of $F(z)$.

Assy 9) Find the inverse z transform of $\frac{z}{z^2-4z-5} = F(z)$

The poles are $z=5$, $z=-1$.

Residue at $z=5$

$$\lim_{z\to 5} \frac{(z-5).z^n}{(z-5)(z+1)} = \frac{5^n}{6}$$

Residue at $z=-1$

$$\lim_{z\to -1} \frac{\cancel{(z+1)}\, z^n}{(z-5)\cancel{(z+1)}} = \frac{(-1)^n}{-6}$$

$f(n)$ = sum of the residues

$$= \frac{1}{6}\left[5^n - (-1)^n\right]$$

---

2) $F(z) = \frac{z}{z^2+2z+2}$

$\alpha = -1+i \quad \beta = -1-i$

are the poles.

Residue at $z=\alpha$ is

$$\lim_{z\to\alpha} \frac{z^n(z-\alpha)}{(z-\alpha)(z-\beta)} = \frac{\alpha^n}{\alpha-\beta}$$

Residue at $z=\beta$ is

$$\lim_{z\to\beta} \frac{z^n(z-\beta)}{(z-\alpha)(z-\beta)} = \frac{\beta^n}{\beta-\alpha}$$

$f(n)$ = sum of residues

$$= \frac{\alpha^n - \beta^n}{\alpha-\beta}$$

$$= \frac{(\sqrt{2})^n}{2i}\left[2i \sin n\theta\right]$$

$$= (\sqrt{2})^n \sin\frac{3n\pi}{4}$$

Assy 10 3) $F(z) = \frac{z^2+z}{(z+2)(z^2+1)}$

The poles are $z=-2, i, -i$

Residue at $z=-2$ is

$$\lim_{z\to -2} \frac{(z+2)(z^2+z)z^n}{(z+2)(z^2+1)} = -\frac{1}{5}(-2)^n$$

Residue at $z=i$ is

$$\lim_{z\to i} \frac{(z-i)z^n(z+1)}{(z+2)(z+i)(z-i)} = \frac{(i)^n(1+i)}{(2+i)2i}$$

Residue at $z=-i$ is

$$\lim_{z\to -i} \frac{(z+i)z^n(z+1)}{(z+2)(z+i)(z-i)} = \frac{(-i)^n(1-i)}{(2-i)(-2i)}$$

$f(n)$ = sum of residues

$$= -\frac{1}{5}(-2)^n + \frac{1}{2i}\frac{(i)^n(3+i)}{5} - \frac{1}{2i}\frac{(-i)^n(3-i)}{5}$$

$$= -\frac{1}{5}(-2)^n + \frac{3}{5}\left[\frac{i^n - (-i)^n}{2i}\right] + \frac{1}{5}\left[\frac{i^{n+1} - (-i)^{n+1}}{2i}\right]$$

$$= -\frac{1}{5}(-2)^n + \frac{3}{5}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{5}\sin\frac{(n+1)\pi}{2}$$

---

3) $F(z) = \frac{z}{(z-4)(z-2)(z-3)}$

The poles are $z = 2, 3, 4$.

Residue at $z=4$

$$\lim_{z\to 4} \frac{(z-4)z^n}{(z-4)(z-2)(z-3)} = \frac{4^n}{2}$$

Residue at $z=2$

$$\lim_{z\to 2} \frac{(z-2)z^n}{(z-4)(z-2)(z-3)} = \frac{2^n}{2}$$

Residue at $z=3$

$$\lim_{z\to 3} \frac{(z-3)z^n}{(z-4)(z-2)(z-3)} = -3^n.$$

$f(n)$ = sum of residues of $F(z).z^{n-1}$ at these poles.

$$= \frac{4^n}{2} + \frac{2^n}{2} - 3^n.$$

1) find $z^{-1}\left[\log\left(\frac{z}{z+1}\right)\right]$

$F(z) = \log\left(\frac{z}{z+1}\right)$ put $z = \frac{1}{y}$

$= -\log(1+y)$

$= -\left[y - \frac{y^2}{2} + \frac{y^3}{3} - \dots\right]$

$= -\frac{1}{z} + \frac{1}{2z^2} \mp \frac{1}{3z^3} + \dots$

$+ \frac{(-1)^n}{n} z^{-n} + \dots$

$$f(n) = \begin{cases} 0 & \text{for } n=0 \\ \frac{(-1)^n}{n} & n \geq 0 \end{cases}$$

---

② $F(z) = \frac{z^2}{(z-2)(z-3)}$

$z^{-1}[F(z)] = z^{-1}\left(\frac{z}{z-2}\right) * z^{-1}\left(\frac{z}{z-3}\right)$

$= 2^n * 3^n$

$= \sum_{K=0}^{n} 2^{n-k} \cdot 3^k$

$= 2^n \sum \left(\frac{3}{2}\right)^k$

$= 2^n \left[1 + \frac{3}{2} + \left(\frac{3}{2}\right)^2 + \dots n \text{ terms}\right]$

$= 2^n \frac{\left[\left(\frac{3}{2}\right)^n - 1\right]}{\frac{3}{2} - 1}$

$= 2^{n+1}\left(\frac{3^{n+1}}{2^{n+1}} - 1\right)$

Hankel Transforms.

The Hankel transform of the function $f(x)$, $-\infty < x < \infty$ is defined as

$$H_n\{f(x)\} = \bar{F}(p) = \int_{-\infty}^{\infty} f(x).\underbrace{x.\, J_n(px)}_{\text{Kernel}}\,dx$$

where $J_n(px)$ is the Bessel function of the first kind of order n and is denoted by

$H_n\{f(x) : p\}$ or $H\{f(x)\}$.

<u>Note</u>: $x J_n(px)$ is called the Kernel of the transformation

<u>Inverse formula for the Hankel transform</u>: If $\bar{F}(p)$ is the Hankel transform of the function $f(r)$.

i.e $\bar{F}(p) = \int_{-\infty}^{\infty} f(r).\, r.\, J_n(pr)\, dr$

then $f(r) = H^{-1}\{\bar{F}(p)\}$

$$= \int_0^{\infty} \bar{F}(p).\, p.\, J_n(pr)\, dp$$

is called the inversion formula

I) <u>Bessel function of first kind</u>

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

II) Recurrence formula for $J_n(x)$

(i) $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

(ii) $x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$

(iii) $2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

(iv) $2n J_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$

(v) $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

(vi) $\frac{d}{dx}\left[x^{n} J_n(x)\right] = x^{n} J_{n-1}(x)$

III) <u>Infinite integrals involving Bessel functions.</u>

(i) $\int_0^{\infty} e^{-ax} J_0(px)\, dx = (a^2+p^2)^{-1/2}$

(ii) $\int_0^{\infty} e^{-ax} J_1(px)\, dx = \frac{1}{p} - \frac{a}{p(a^2+p^2)^{1/2}}$

(iii) $\int_0^{\infty} x e^{-ax} J_0(px)\, dx = a(a^2+p^2)^{-3/2}$

(iv) $\int_0^{\infty} x e^{-ax} J_1(px)\, dx = p(a^2+p^2)^{-3/2}$

(v) $\int_0^{\infty} \frac{1}{x} e^{-ax} J_1(px)\, dx = \frac{(a^2+p^2)^{1/2} - a}{p}$

---

1) Find the Hankel transform of (i) e^{-x} (ii) $\frac{e^{-x}}{x}$ (iii) $\frac{e^{-ax}}{x}$ taking $x J_0(px)$ as the kernel of the transformation.

(i) $H\{e^{-x}\} = \int_0^{\infty} x e^{-x} J_0(px)\, dx$

$= (1+p^2)^{-3/2} = a(a^2+p^2)^{-3/2}$

(ii) $H\left\{\frac{e^{-x}}{x}\right\} = \int_0^{\infty} \frac{e^{-x}}{x}.\, x.\, J_0(px)\, dx$

$= (1+p^2)^{-1/2} = (a^2+p^2)^{-1/2}$

(iii) $H\left\{\frac{e^{-ax}}{x}\right\} = \int_0^{\infty} \frac{e^{-ax}}{x}\, x\, J_0(px)\, dx$

$= (a^2+p^2)^{-1/2}$

2) Find the Hankel transform of e^{-ax} taking $x J_0(px)$ as the kernel of the transform

$$H\{e^{-ax}\} = a(a^2+p^2)^{-3/2}$$

3) Find the Hankel transform of $f(x) = \begin{cases} 1 & 0<x<a \quad n=0 \\ 0 & x>a \quad n=0 \end{cases}$

Soln:

$$H_n\{f(x)\} = \int_0^\infty x . f(x) . J_n(px)dx$$

$$= \int_0^a x . 1 . J_0(px)dx \quad —(1)$$

From Recurrence formula for Bessel fn we have $\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$

$$\frac{d}{dx}[x . J_1(x)] = x . J_0(x)$$

writing px for x, we have

$$\frac{1}{p} . \frac{d}{dx}\{px . J_1(px)\} = px J_0(px)$$

~~From (1)~~ Integrating on both sides

$$\int_0^a x J_0(px)dx = \left[\frac{x}{p} . J_1(px)\right]_0^a$$

$$H\{f(x)\} = \frac{a}{p} . J_1(pa)$$

4) $H\{x^2 e^{-x}\}$ taking $x J_1(px)$ as the kernel.

$$= \int_0^\infty x^{-2} e^{-x} . x . J_1(px)\, dx .$$

$$= \int_0^\infty x^{-1} . e^{-x} . J_1(px)\, dx .$$

$$= \cancel{\frac{(1+p^2)^{-1} - 1}{p}}$$

$$\boxed{\begin{aligned} px &= x . \\ p . dx &= dx . \end{aligned}}$$

6) $H\left\{\frac{\sin ax}{x}\right\}$ take $x J_0(px)$ as kernel.

$$= \int_0^\infty \frac{\sin ax}{x} . x J_0(px)\, dx .$$

$$= - I.P. \text{ of } \int_0^\infty e^{-iax} J_0(px)\, dx .$$

$$= - I.P. \text{ of } (i^2a^2 + p^2)^{-1/2}$$

$$= - I.P. \text{ of } (p^2 - a^2)^{-1/2}$$

$$= \begin{cases} 0 & \text{if } p>a \text{ it is real } \boxed{I.P = 0} . \\ (a^2-p^2)^{-1/2} & \text{if } 0<p<a . \end{cases}$$

7) $H\{e^{-ax}\}$ taking $x J_1(px)$ as the Kernel.

$$= \int_0^\infty e^{-ax} . x J_1(px)\, dx$$

$$= p(a^2+p^2)^{-3/2}$$

8) $H^{-1}\left\{\frac{e^{-ap}}{p}\right\}$ when $n=1$.

Applying inversion formula

$$= \int_0^\infty \frac{e^{-ap}}{p} . p . J_1(px)\, dp$$

$$= \int_0^\infty e^{-ap} J_1(px)\, dp$$

$$= \frac{1}{x} - \frac{a}{x(a^2+x^2)^{1/2}}$$

9) $H^{-1}\{p^{-2} e^{-ap}\}$ taking $n=1$.

$$= \int_0^\infty p^{-2} e^{-ap} . p \, J_1(px)\, dp$$

$$= \int_0^\infty \frac{1}{p} . e^{-ap} . J_1(px)\, dp$$

$$= \frac{(a^2+x^2)^{1/2} - a}{x}$$

$(-1)^{1/2} = \sqrt{-1} = i$

Note: $(p^2-a^2)^{-1/2} = \frac{1}{(p^2-a^2)^{1/2}}$

$$= \frac{1}{[-(a^2-p^2)]^{1/2}} = \frac{1}{i(a^2-p^2)^{1/2}} = -i(a^2-p^2)^{-1/2}$$

(10) Ass 5/6 $H^{-1}\{\bar{F}(p)\}$ if $\bar{F}(p)=\frac{e^{-ap}}{p}$

where $\bar{F}(p)$ is the Hankel transform of $f(x)$ and $n=0$.

Applying inversion formula

$$f(x)=H^{-1}\{\bar{F}(p)\}$$

$$=\int_0^\infty \bar{F}(p).\,p\,J_0(px)\,dp$$

$$=\int_0^\infty \frac{e^{-ap}}{\cancel{p}}.\cancel{p}.J_0(px)\,dp$$

$$=(a^2+x^2)^{-1/2}$$

---

11) Find the Hankel transform of $\frac{1}{r}$ and then apply the inversion formula to get the original function.

Let $f(r)=\frac{1}{r}$

$$H\{f(r)\}=\bar{F}(p)=\int_0^\infty \frac{1}{r}.r.J_0(pr)\,dr$$

$$=\int_0^\infty e^{-ap}.J_0(pr)\,dr \quad (\because a=0)$$

$$=(a^2+p^2)^{-1/2}=(p^2)^{-1/2}$$

$$=\frac{1}{p}.$$

Now applying the inversion formula.

$$f(r)=\int_0^\infty \bar{F}(p).\,p.\,J_0(pr)\,dp=\boxed{\int_0^\infty \frac{1}{\cancel{p}}\cancel{p}.J_0(pr)\,dp}$$

$$=\int_0^\infty J_0(pr)\,dp=\frac{1}{r}.$$

Ass 5/4 12) Find the Hankel transform of

$$f(x)=\begin{cases} x^n & 0<x<a \\ 0 & x>a.\end{cases}$$

$n>-1$, taking $x\,J_n(px)$ as the kernel.

Soln: $H\{f(x)\}=\int_0^\infty f(x).\,x\,J_n(px)\,dx.$

$$=\int_0^a x^n.\,x\;J_n(px)\,dx.$$

$$=\int_0^a x^{n+1}.\,J_n(px)\,dx. \quad \text{—(1)}$$

Now replacing x by px and n by $n+1$ in the formula

$$\frac{d}{dx}[x^n J_n(x)]=x^n J_{n-1}(x)$$

we get

$$\frac{1}{p}\frac{d}{dx}\left[x^{n+1}J_{(n+1)}(px)\right]=x^{n+1}J_n(px)$$

From (1) we have

$$H\{f(x)\}=\int_0^a \frac{1}{p}.\frac{d}{dx}\left[x^{n+1}.J_{n+1}(px)\right]dx$$

$$=\frac{1}{p}\left[x^{n+1}.\,J_{n+1}(px)\right]_0^a$$

$$=\frac{1}{p}\left[a^{n+1}.\,J_{n+1}(pa)\right]$$

HT of derivatives of a function

The HT of order n of the function $f(x)$ is given by

$$\tilde{f}_n(p)=\int_0^\infty x\,f(x)\,J_n(px)\,dx.$$

If $\tilde{f}'_n(p)$ is the HT of $\frac{df}{dx}$ then

$$\tilde{f}'_n(p)=\int_0^\infty x.\frac{df}{dx}\,J_n(px)\,dx.$$

Formulae.

1) $\tilde{f}'_1(p)=-p\tilde{f}_0(p)$

2) $\tilde{f}'_2(p)=-p\left[\frac{3}{4}\tilde{f}_1(p)-\frac{1}{4}\tilde{f}_3(p)\right]$

3) $\tilde{f}'_3(p)=-p\left[\frac{2}{3}\tilde{f}_2(p)-\frac{1}{3}\tilde{f}_4(p)\right]$

4) Find the Hankel transform of

Ans/ $f(x) = \begin{cases} a^2 - x^2, & 0 < x < a, \quad n=0 \\ 0 & x > a \quad n=0 \end{cases}$

$$\boxed{\begin{aligned} \text{Kernel} &= x\, J_n(px) \\ &= x \cdot J_0(px) \end{aligned}}$$

Solu: $H_n\{f(x)\} = \int_0^{\infty} x\, f(x)\, J_n(px)\, dx = \int_0^a x(a^2 - x^2)\, J_0(px)\, dx.$

$$= a^2 \int_0^a x\, J_0(px)\, dx - \int_0^a x^3 J_0(px)\, dx. \quad \text{—} (1)$$

Replacing px for x and $n=1$ in the formula $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$ — (2)

we get $\frac{1}{p} \cdot \frac{d}{dx}\{x J_1(px)\} = x \cdot J_0(px)$

consider 2nd integral

$$\int_0^a x^3 J_0(px)\, dx = \int_0^a \underbrace{x^2}_{u} \cdot \underbrace{\frac{1}{p} \cdot \frac{d}{dx}\{x J_1(px)\}}_{v}\, dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx$$

$$= \left[x^2 \cdot \frac{x}{p} J_1(px)\right]_0^a - \frac{1}{p}\int_0^a (2x)\, x \cdot J_1(px)\, dx$$

$$= \frac{a^3}{p} J_1(pa) - \frac{2}{p}\int_0^a x^2 J_1(px)\, dx \quad \text{—} (3)$$

writing px for x and $n=2$ in (2)

$$\frac{1}{p} \cdot \frac{d}{dx}\left[x^2 J_2(px)\right\} = x^2 J_1(px)$$

From (3).

$$\therefore \int_0^a x^3 J_0(px)\, dx = \frac{a^3}{p} J_1(pa) - \frac{2}{p}\int_0^a \frac{1}{p} \cdot \frac{d}{dx}\{x^2 J_2(px)\}\, dx$$

$$= \frac{a^3}{p} \cdot J_1(pa) - \frac{2}{p^2} \cdot a^2 J_2(pa) \quad \text{—} (4)$$

We know $2n\, J_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$

writing pa for x and $n=1$ in the above.

$$2 \cdot J_1(pa) = pa\left[J_0(pa) + J_2(pa)\right] \Rightarrow \frac{2}{pa} J_1(pa) = J_0(pa) + J_2(pa)$$

From (4) $\int_0^a x^3 J_0(px)\, dx = \frac{a^3}{p} J_1(pa) - \frac{2a^2}{p^2}\left[\frac{2}{pa} J_1(pa) - J_0(pa)\right]$

From (1),

$$H_n\{f(x)\} = a^2 \cdot \cancel{\frac{a}{p} J_1(pa)} - \cancel{\frac{a^3}{p} J_1(pa)} + \frac{4a}{p^3} J_1(pa) - \frac{2a^2}{p^2} J_0(pa)$$

$$= \frac{4a}{p^3} \cdot J_1(pa) - \frac{2a^2}{p^2} J_0(pa)$$

$$\boxed{\text{Note: } \int_0^a x\, J_0(px)\, dx = \frac{a}{p} J_1(pa) \quad \text{proved.}}$$

Ass 5 1) Find the Hankel transform of $\frac{df}{dx}$ when $f = \frac{e^{-ax}}{x}$ and $n=1$.

Sol:
$$H\left\{\frac{df}{dx}\right\} = \int_0^\infty x \cdot \frac{df}{dx} \cdot J_1(px)\,dx$$
$$= \tilde{f}_1'(p) = -p\tilde{f}_0(p)$$
$$= -p\int_0^\infty x f(x)\,J_0(px)\,dx$$
$$= -p\int_0^\infty \cancel{x} \cdot \frac{e^{-ax}}{\cancel{x}} \cdot J_0(px)\,dx.$$
$$= -p\,(a^2+p^2)^{-1/2}$$

2) Find the Hankel transform of $\frac{\partial^2 f}{\partial t^2}$ where f is a function of x and t.

Sol
$$H\left\{\frac{\partial^2 f}{\partial t^2}\right\} = \int_0^\infty x \cdot \frac{\partial^2 f}{\partial t^2}\, J_n(px)\,dx$$
$$= \frac{d^2}{dt^2}\int_0^\infty x \cdot f(x,t)\, J_n(px)\,dx.$$
$$= \frac{d^2}{dt^2}\bar{f}(p,t)$$

3) Hankel transform of Ass 5 3
$$\frac{d^2f}{dx^2} + \frac{1}{x}\cdot\frac{df}{dx} - \frac{n^2}{x^2}f$$

Sol we have
$$H\left\{\frac{d^2f}{dx^2}\right\} = \int_0^\infty \underbrace{\frac{d^2f}{dx^2}}_{v} \cdot \underbrace{x\,J_n(px)}_{u}\,dx$$
Integrating by parts taking $x\,J_n(px)$ as first function.
$$= \left[x \cdot J_n(px)\cdot\frac{df}{dx}\right]_0^\infty - \int_0^\infty \frac{d}{dx}\left[x J_n(px)\right]\cdot\frac{df}{dx}\,dx$$
$$= -\int_0^\infty \frac{df}{dx}\left[J_n(px) + px\,J_n'(px)\right]dx$$
(Assuming that $x f'(x) \to 0$ when $x \to 0$ or when $x \to \infty$)

$$\int_0^\infty\left(\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx}\right)x\,J_n(px)\,dx$$
$$= -p\int_0^\infty \frac{df}{dx}\cdot x\,J_n'(px)\,dx$$
Integrating the integral on the right by parts taking $x\,J_n'(px)$ as first function
$$= -p\left[x\,J_n'(px)\,f(x)\right]_0^\infty$$
$$+ p\int_0^\infty f(x)\,\frac{d}{dx}\left\{x\,J_n'(px)\right\}dx$$
$$= p\int_0^\infty f(x)\cdot\frac{d}{dx}\left\{x\,J_n'(px)\right\}dx \quad (1)$$
(assuming that $xf(x) \to 0$ when $x \to 0$ or when $x \to \infty$)

Now since $J_n(x)$ satisfies the Bessel's D. E.
$$\frac{d}{dx}\left(x\cdot\frac{dy}{dx}\right) + \left(1-\frac{n^2}{x^2}\right)xy = 0$$
$$\frac{d}{dx}\left(x\cdot\frac{d}{dx}J_n(x)\right) + \left(1-\frac{n^2}{x^2}\right)x\,J_n(x) = 0$$
Replace x by px
$$\frac{1}{p}\frac{d}{dx}\left[px\,J_n'(px)\right] = -\left(p^2-\frac{n^2}{x^2}\right)\frac{x}{p}\,J_n(px)$$
$$\frac{d}{dx}\left[x\cdot J_n'(px)\right] = -\left(p^2-\frac{n^2}{x^2}\right)\frac{x}{p}\,J_n(px)$$
Hence from (1) we have
$$\int_0^\infty\left(\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx}\right)x\,J_n(px)\,dx$$
$$\cancel{= -p\int_0^\infty f(x)\cdot x\,J_n(px)\,dx}$$
$$= -\cancel{p}\int_0^\infty f(x)\left(p^2-\frac{n^2}{x^2}\right)\frac{x}{\cancel{p}}\,J_n(px)\,dx$$
$$\int_0^\infty\left(\frac{d^2f}{dx^2} + \frac{1}{x}\cdot\frac{df}{dx} - \frac{n^2}{x^2}f\right)x\,J_n(px)\,dx$$
$$= -p^2\int_0^\infty f(x)\cdot x\,J_n(px)\,dx$$
$$= -p^2\,\bar{f}_n(p)$$

calculation

$$\therefore \int_0^\infty \frac{d^2f}{dx^2}\cdot \underbrace{x\, J_n(px)\,dx}_{u} = \left[x\cdot J_n(px)\,\frac{df}{dx}\right]_0^\infty - \int_0^\infty \frac{d}{dx}\left[x\cdot J_n(px)\right]\frac{df}{dx}\,dx$$

$$= 0 - \int_0^\infty \left\{\left[1\cdot J_n(px) + x\cdot J_n'(px)\cdot p\right]\frac{df}{dx}\right\}dx$$

$$= -\int_0^\infty \frac{df}{dx}\cdot J_n(px)\,dx - p\int_0^\infty \frac{df}{dx}\cdot x\cdot J_n'(px)\,dx$$

$$\int_0^\infty \left[\frac{d^2f}{dx^2}\cdot x\cdot J_n(px) + \frac{df}{dx}\cdot J_n(px)\right]dx = -p\int_0^\infty \frac{df}{dx}\cdot \underbrace{x\cdot J_n'(px)}_{u}\,dx.$$

$$\int_0^\infty \left(\frac{d^2f}{dx^2} + \frac{1}{x}\cdot\frac{df}{dx}\right)x\, J_n(px)\,dx$$

$$= -p\left[x\,J_n'(px)\cdot f(x)\right]_0^\infty + p\int_0^\infty f(x)\cdot\frac{d}{dx}\left\{x\,J_n'(px)\right\}dx.$$

$$= 0 + p\int_0^\infty f(x)\cdot\frac{d}{dx}\left\{x\,J_n'(px)\right\}dx.$$

$$= -\cancel{p}\int_0^\infty f(x)\left(p^2 - \frac{n^2}{x^2}\right)\cdot\frac{x}{\cancel{p}}\,J_n(px)\,dx$$

$$= -p^2\int_0^\infty f(x)\cdot x\,J_n(px)\,dx + \int_0^\infty \frac{n^2}{x^2}\,f(x)\cdot x\,J_n(px)\,dx.$$

$$\int_0^\infty \left(\frac{d^2f}{dx^2} + \frac{1}{x}\cdot\frac{df}{dx} - \frac{n^2}{x^2}\cdot f\right)x\,J_n(px)\,dx = -p^2\,\tilde{f}_n(p)$$

Deductions:

1) putting $n=0$, we get

$$\int_0^\infty \left(\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right)x J_0(px)\,dx = -p^2\tilde{f}_0(p)$$

where $\tilde{f}_0(p)$ is HT of $f(x)$ of zero order.

2) putting $n=1$,

$$\int_0^\infty \left(\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}-\frac{1}{x^2}\right)x J_1(px)\,dx$$

$$= -p^2\tilde{f}_1(p)$$

---

4) Evaluate $\int_0^\infty r\left(\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}\right)J_0(pr)\,dr$

where $f(r) = \frac{e^{-ar}}{r}$.

Sol

$$\tilde{f}_0(p) = \int_0^\infty r f(r) J_0(pr)\,dr$$

$$= \int_0^\infty \cancel{r}\cdot\frac{e^{-ar}}{\cancel{r}}\cdot J_0(pr)\,dr.$$

$$= \frac{1}{\sqrt{a^2+p^2}}$$

From deduction (1) above we have

$$\int_0^\infty r\left(\frac{d^2f}{dr^2}+\frac{1}{r}\cdot\frac{df}{dr}\right)J_0(pr)\,dr$$

$$= -p^2\tilde{f}_0(p) = -\frac{p^2}{\sqrt{a^2+p^2}}$$

Finite Hankel transform.

of the function $f(r)$, which satisfies Dirichlet's conditions in the closed interval $0\le r\le a$, is defined by the definite integral

$$\tilde{f}(p_i) = \int_0^a r f(r) J_n(p_i r)\,dr$$

where p_i is a root of the transcendental eqn. $J_n(ap_i)=0$ and at any point of the closed interval $0\le r\le a$ at which the function $f(r)$ is continuous, we have

$$f(r) = \frac{2}{a^2}\sum_i \tilde{f}(p_i)\cdot\frac{J_n(rp_i)}{[J_n'(ap_i)]^2}$$

where the sum is considered over all the positive roots of $J_n(ap_i)=0$

$$\bar{f}(p) = \int_0^\infty f(x)\cdot x\cdot J_n(px)\,dx.$$

Ans 6

1) Show that $\int_0^a r J_0(pr)\,dr = \frac{a}{p} J_1(ap)$

from recurrence formula for Bessel fn we have $\frac{d}{dr}\left[r^n J_n(r)\right] = r^n J_{n-1}(r)$

We know $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

Replacing $n=1$ and r by pr

$$\frac{1}{p}\frac{d}{dr}\left[pr J_1(pr)\right] = pr J_0(pr)$$

$$\frac{1}{p}\frac{d}{dr}\left[r J_1(pr)\right] = r J_0(pr)$$

$$\int_0^a r J_0(pr)\,dr = \frac{1}{p}\left[r J_1(pr)\right]_0^a$$

$$= \frac{a}{p} J_1(ap).$$

2) Find HT of c if $r J_0(p_i r)$ be the the Kernel.

Sol: If $f(r) = c$ then

$$\tilde{f}(p_i) = \int_0^a \overset{f(r)}{r}\cdot J_0(p_i r)\,dr$$

$$= c\int_0^a r J_0(p_i r)\,dr.$$

$$= \frac{ca}{p_i} J_1(ap_i).$$

3) Find the HT of r^n $(n > -1)$ if $r J_n(p_i r)$ is the Kernel

Sol. $f(r) = r^n$

$$\tilde{f}(p_i) = \int_0^a r^n \cdot r J_n(p_i r)\,dr$$

$$= \int_0^a r^{n+1} J_n(p_i r)\,dr$$

Replacing r by $p_i r$ and n by $n+1$ in $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

$$\frac{1}{p_i}\frac{d}{dr}\left[r^{n+1} J_{n+1}(p_i r)\right] = r^{n+1} J_n(p_i r)$$

$$\tilde{f}(p_i) = \frac{1}{p_i}\int_0^a \frac{d}{dr}\left(r^{n+1} J_{n+1}(p_i r)\right)dr$$

$$= \frac{a^{n+1}}{p_i} J_{n+1}(ap_i)$$

Ans 6 Assign 6

4) Find the HT of $a^2 - r^2$ if $r J_0(pr)$ is the kernel

$$H_0(a^2 - r^2) = \int_0^a (a^2 - r^2)\cdot r\cdot J_0(pr)\,dr$$

$$= a^2\int_0^a r J_0(pr)\,dr - \int_0^a r^3 J_0(pr)\,dr$$

$$= a^2\cdot\frac{a}{p} J_1(ap) - \frac{a^2}{p^2}\left[2 J_0(pa) + \left(ap - \frac{4}{ap}\right) J_1(pa)\right]$$

$$= \frac{4a}{p^3} J_1(ap) - \frac{2a^2}{p^2} J_0(pa).$$

Hankel transform of Derivatives

Hankel transform of $\frac{df}{dx}$.

$$H_n\left(\frac{df}{dx}\right) = \int_0^a \frac{df}{dx}\cdot x\cdot J_n(px)\,dx$$

where p is the root of the equation $J_n(pa) = 0$.

Integrating by parts.

$$= \left[f(x)\cdot x\, J_n(px)\right]_0^a - \int_0^a f(x)\cdot\frac{d}{dx}\{x\,J_n(px)\}\,dx$$

$$= -\int_0^a f(x)\cdot\frac{d}{dx}\{x\cdot J_n(px)\}\,dx \quad \text{—(1)}$$

since $J_n(pa) = 0$

$$\frac{d}{dx}\{x\cdot J_n(px)\} = x\cdot p\cdot J_n'(px) + J_n(px) \quad \text{—(2)}$$

and from recurrence formula for Bessel functions, we have

$$2\,J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

and $2n\,J_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$

writing px for x, we have

$$J_n'(px) = \frac{1}{2}\left[J_{n-1}(px) - J_{n+1}(px)\right]$$

$$J_n(px) = \frac{px}{2n}\left[J_{n-1}(px) + J_{n+1}(px)\right]$$

From (2) we have

$$\frac{d}{dx}\{x\,J_n(px)\} = \frac{px}{2}\left[J_{n-1}(px) - J_{n+1}(px)\right] + \frac{px}{2n}\left[J_{n-1}(px) + J_{n+1}(px)\right]$$

$$= \frac{p}{2n}\left[(n+1)\,x\,J_{n-1}(px) - (n-1)\cancel{x}J_{n+1}(px)\right]$$

From (1) we have

$$H_n\left\{\frac{df}{dx}\right\} = -\frac{p}{2n}\int_0^a\left[(n+1)\,x\,f(x)\,J_{n-1}(px) - (n-1)\,x\,f(x)\,J_{n+1}(px)\right]dx$$

$$= \frac{p}{2n}\left[-\int_0^a (n+1)\,x\,f(x)\,J_{n-1}(px)\,dx + \int_0^a (n-1)\,x\,f(x)\,J_{n+1}(px)\,dx\right]$$

$$= \frac{p}{2n}\left[-(n+1)\,H_{n-1}\{f(x)\} + (n-1)\,H_{n+1}\{f(x)\}\right] \quad \text{—(3)}$$

Cor: If $n=1$, then we have.

$$H_1\left\{\frac{df}{dx}\right\} = -p\,H_0\{f(x)\}.$$

Ans 10. Hankel transform of $\frac{d^2f}{dx^2} + \frac{1}{x}\cdot\frac{df}{dx}$ when p is the root of the equation $J_n(ap) = 0$.

Sol: $H_n\left\{\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx}\right\}$

$$= \int_0^a x\left\{\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx}\right\} J_n(px)\,dx.$$

$$= H_n\left\{\frac{d^2f}{dx^2}\right\} + \int_0^a \frac{df}{dx}\,J_n(px)\,dx. \quad \text{—(1)}$$

Replacing f by $\frac{df}{dx}$ in (3),

$$H_n\left\{\frac{d^2f}{dx^2}\right\} = \frac{p}{2n}\left[-(n+1)\,H_{n-1}\left\{\frac{df}{dx}\right\} + (n-1)\,H_{n+1}\left\{\frac{df}{dx}\right\}\right] \quad \text{—(2)}$$

From recurrence formula for Bessel functions, we have

$$2n\,J_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

Replacing x by px, we have

$$J_n(px) = \frac{px}{2n}\left[J_{n-1}(px) + J_{n+1}(px)\right]$$

$$\therefore \int_0^a \frac{df}{dx}\,J_n(px)\,dx$$

$$= \int_0^a \frac{df}{dx}\cdot\frac{px}{2n}\left[J_{n-1}(px) + J_{n+1}(px)\right]dx$$

$$= \frac{p}{2n}\int_0^a \left[x\frac{df}{dx} J_{n-1}(px) + x\cdot\frac{df}{dx} J_{n+1}(px)\right] dx$$

$$= \frac{p}{2n}\left[H_{n-1}\left\{\frac{df}{dx}\right\} + H_{n+1}\left\{\frac{df}{dx}\right\}\right] \quad \text{—(3)}$$

Substituting (2) and (3) in (1)

$$H_n\left\{\frac{d^2f}{dx^2} + \frac{1}{x}\cdot\frac{df}{dx}\right\}$$

$$= \frac{p}{2n}\left[-n H_{n-1}\left\{\frac{df}{dx}\right\} + n H_{n+1}\left\{\frac{df}{dx}\right\}\right]$$

or $H_n\left\{\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx}\right\}$

$$\boxed{= \frac{p}{2}\left[-H_{n-1}\left\{\frac{df}{dx}\right\} + H_{n+1}\left\{\frac{df}{dx}\right\}\right]}$$

Hankel transform of Ans 5/1.

$\frac{d^2f}{dx^2} + \frac{1}{x}\cdot\frac{df}{dx} - \frac{n^2}{x^2} f(x)$ when p is a root of the equation $J_n(ap) = 0$.

Sol: $H_n\left\{\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx} - \frac{n^2}{x^2} f(x)\right\}$

$$= \int_0^a \left[\frac{d^2f}{dx^2} + \frac{1}{x}\cdot\frac{df}{dx} - \frac{n^2}{x^2} f(x)\right]\cdot x J_n(px)\,dx$$

$$= \int_0^a \frac{d^2f}{dx^2}\cdot x\cdot J_n(px)\,dx + \int_0^a \frac{df}{dx}\cdot J_n(px)\,dx$$

$$- n^2\int_0^a \frac{1}{x} f(x)\, J_n(px)\,dx$$

$$= \left[\frac{df}{dx}\cdot x J_n(px)\right]_0^a - \int_0^a \frac{df}{dx}\cdot\frac{d}{dx}\{x J_n(px)\}$$

$$+ \int_0^a \frac{df}{dx}\cdot J_n(px)\,dx - n^2\int_0^a \frac{1}{x} f(x) J_n(px)\,dx$$

(Integrating first integral on RHS by parts)

$$= 0 - \int_0^a \frac{df}{dx}\cdot\{px J_n'(px) + J_n(px)\}\,dx$$

$$+ \int_0^a \frac{df}{dx} J_n(px)\,dx$$

$$- n^2\int_0^a \frac{1}{x}\cdot f(x)\cdot J_n(px)\,dx$$

since $J_n(pa) = 0$

$$= -p\int_0^a \frac{df}{dx}\cdot x J_n'(px)\,dx$$

$$- n^2\int_0^a \frac{1}{x}\cdot f(x)\ J_n(px)\,dx$$

$$= -p\left[f(x)\cdot x\cdot J_n'(px)\right]_0^a$$

$$+ p\int_0^a f(x)\cdot\frac{d}{dx}\{x\cdot J_n'(px)\}\,dx$$

$$- n^2\int_0^a \frac{1}{x}\cdot f(x)\cdot J_n(px)\,dx$$

$$= -pa\, f(a)\, J_n'(ap)$$

$$+ p\int_0^a f(x)\{px J_n''(px) + J_n'(px)\}\,dx$$

$$- n^2\int_0^a \frac{1}{x}\cdot f(x)\, J_n(px)\,dx \quad \text{—(1)}$$

Now $J_n(x)$ is the solution of Bessel D.E.

$$x^2\frac{d^2y}{dx^2} + x\cdot\frac{dy}{dx} + (x^2 - n^2)\,y = 0.$$

$$p^2x^2 J_n''(px) + px\cdot J_n'(px) + (p^2x^2 - n^2)\, J_n(px) = 0.$$

(Replacing x by px), ~~we have~~

$$p^2x^2 J_n''(px) + J_n'(px)$$

$$= px\left(\frac{n^2}{p^2x^2} - 1\right) J_n(px)$$

putting in (1) we have

$$LHS = -pa\, f(a)\, J_n'(ap)$$

$$+ p\int_0^a f(x)\cdot\left\{px\left(\frac{n^2}{p^2x^2} - 1\right) J_n(px)\right\} dx$$

$$- n^2\int_0^a \frac{1}{x}\cdot f(x)\, J_n(px)\,dx.$$

$$= -pa\, f(a)\, J_n'(ap) - p^2\int_0^a f(x)\cdot x J_n(px)\,dx$$

$$= -pa\, f(a)\, J_n'(ap) - p^2 H_n\{f(x)\}.$$

put $a = 1$.

$$LHS = -p f(1)\, J_n'(p) - p^2 H_n\{f(x)\}$$

But $J_0'(p) = -J_1(p)$.

Prob Assignment 5 problems.

1) Find the Hankel transform of r^{n-1} $(n>0)$ if $r.\,J_{n-1}(p_i r)$ is the kernel of the transform.

Soln. If $f(r) = r^{n-1}$ then

$$\tilde{f}(p_i) = \int_0^a r^{n-1}.\, r.\, J_{n-1}(p_i r)\, dr$$

$$= \int_0^a r^n J_{n-1}(p_i r)\, dr$$

Now replacing r by $p_i r$ in recurrence formula for Bessel fns. we have.

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

$$\frac{d}{dr}\left[r^n J_n(r)\right] = r^n J_{n-1}(r)$$

$$\frac{1}{p_i}\cdot\frac{d}{dr}\left[r^n J_n(p_i r)\right] = r^n J_{n-1}(p_i r)$$

$$\bar{f}(p_i) = \int_0^a \frac{1}{p_i}\frac{d}{dr}\left[r^n J_n(p_i r)\right] dr$$

$$= \frac{1}{p_i}\left[r^n J_n(p_i r)\right]_0^a$$

$$= \frac{a^n}{p_i}\cdot J_n(p_i a)$$

2) Show that

$$\int_0^a r^3 J_0(pr)\, dr = \frac{a^2}{p^2}\left[2J_0(pa) + \left(ap - \frac{4}{ap}\right) J_1(pa)\right]$$

Ans 5/6

Sol From recurrence formula for Bessel fn, we have.

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

writing $n=1$ and $x = pr$

$$\frac{1}{p}\frac{d}{dr}\left[r J_1(pr)\right] = r J_0(pr) \quad (1)$$

$$\int_0^a r^3 J_0(pr)\, dr.$$

$$= \int_0^a r^2.\frac{1}{p}\cdot\frac{d}{dr}\{r J_1(pr)\}\, dr.$$

$$= \left[\frac{r^2}{p}.\, r J_1(pr)\right]_0^a - \int_0^a \frac{2r}{p}.\, r J_1(pr)\, dr$$

$$= \frac{a^3}{p} J_1(pa) - \frac{2}{p}\int_0^a \frac{1}{p}\cdot\frac{d}{dr}\left(r^2 J_2(pr)\right) dr$$

$$= \frac{a^3}{p} J_1(pa) - \frac{2}{p^2}\left[r^2 J_2(pr)\right]_0^a$$

$$= \frac{a^3}{p} J_1(pa) - \frac{2a^2}{p^2} J_2(pa).$$

From recurrence formula for Bessel fn.

$$\frac{2n}{x}\cdot J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

writing $n=1$, $x = pa$ we have

$$J_2(pa) = \frac{2n}{pa} J_1(pa) - J_0(pa)$$

$$\therefore \int_0^a r^3 J_0(pr)\, dr = \frac{a^3}{p} J_1(pa) - \frac{2a^2}{p^2}\left[\frac{2}{pa} J_1(pa) - J_0(pa)\right]$$

$$= \frac{a^2}{p^2}\left[2J_0(pa) + \left(ap - \frac{4}{ap}\right) J_1(pa)\right]$$

Applications of Hankel transform in Initial and Boundary value problems :- In a D.E. in which a variable ranges from 0 to ∞, this variable can be excluded with the help of Hankel transforms.

Assignment. 9

1) The vibrations of a very large membrane are governed by the eqn

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2}, \quad r \geq 0,\ t \geq 0$$

with $U = f(r)$, $\frac{\partial U}{\partial r} = g(r)$ when $t = 0$. Show that for $t > 0$

$$U(r,t) = \int_0^\infty p\, \tilde{f}(p) \cos(pct)\, J_0(pr)\, dp + \frac{1}{c}\int_0^\infty \tilde{g}(p) \sin(pct)\, J_0(pr)\, dp$$

where $\tilde{f}(p)$ and $\tilde{g}(p)$ are the zero order Hankel transforms respectively of $f(r)$ and $g(r)$.

Soln:- Taking the Hankel transform for $n=0$ of both the sides of the given D.E. and the given conditions, we have

$$\int_0^\infty \left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right) r\, J_0(pr)\, dr = \frac{1}{c^2}\int_0^\infty \frac{\partial^2 U}{\partial t^2}\, r\, J_0(pr)\, dr$$

$$-p^2 \tilde{U} = \frac{1}{c^2}\frac{d^2}{dt^2}\int_0^\infty U\, r\, J_0(pr)\, dr$$

where

$$\tilde{U} = \int_0^\infty U\, r\, J_0(pr)\, dr$$

$$-p^2\tilde{U} = \frac{1}{c^2}\frac{d^2\tilde{U}}{dt^2}$$

$$\frac{d^2\tilde{U}}{dt^2} + c^2p^2\tilde{U} = 0. \quad \text{—— (1)}$$

Also when $t = 0$

$$\tilde{U} = \int_0^\infty f(r)\, r\, J_0(pr)\, dr = \tilde{f}(p) \quad \text{—— (2)}$$

and $\int_0^\infty \frac{\partial U}{\partial r}\, r\, J_0(pr)\, dr = \int_0^\infty g(r)\, r\, J_0(pr)\, dr$

$$\frac{d\tilde{U}}{dt} = \tilde{g}(p)$$

$$\boxed{H\left\{\frac{d^2 f}{dx^2} + \frac{1}{x}\frac{df}{dx}\right\} = -p^2\tilde{f}_0(p)}$$

A.E. of (1) is $m^2 = -c^2p^2$

$m = \pm icp$

$$\tilde{U} = A\cos(pct) + B\sin(pct) \quad \text{—— (3)}$$

when $t = 0$ from (1) and (2)

$\tilde{f}(p) = A.$

$$\frac{d\tilde{U}}{dt} = \tilde{g}(p)$$

$$-Apc\sin(pct) + Bpc\cos(pct) = \tilde{g}(p)$$

when $t=0$, $B = \frac{\tilde{g}(p)}{pc}$

from (3),

$$\tilde{U} = \tilde{f}(p)\cos(pct) + \frac{\tilde{g}(p)}{pc}\sin(pct)$$

~~Applying inversion formula~~

$$U(r,t) = \int_0^\infty p\, \tilde{f}(p)\cos(pct)\, J_0(pr)\, dp + \frac{1}{c}\int_0^\infty \tilde{g}(p)\sin(pct)\, J_0(pr)\, dp.$$