



J.B. INSTITUTE OF ENGINEERING AND TECHNOLOGY

(UGC AUTONOMOUS)

Bhaskar Nagar, Moinabad Mandal, R.R. District, Hyderabad -500075

DEPARTMENT OF CIVIL ENGINEERING

ENGINEERING MECHANICS

LECTURE NOTES R18

B. TECH

I YEAR - II SEM (Sec- A&B)

Academic Year 2020-21

Prepared & Compiled by

MR. Shiva Prasad

ASSISTANT PROFESSOR, DEPARTMENT OF CE

J.B.I.E.T

**Bhaskar Nagar, Yenkapally(V), Moinabad(M),
Ranga Reddy(D), Hyderabad - 500 075, Telangana, India**

UNIT – I

KINEMATICS OF PARTICLES- RECTILINEAR MOTION

Motion in a plane: Introduction to polar coordinates

So far we have discussed equilibrium of bodies i.e. we have concentrated only on statics. From this lecture onwards we learn about the motion of particles and composite bodies and how it is affected by the forces applied on the system. Thus we are now starting study of dynamics.

When we describe the motion of a particle, we specify it by giving its position and velocity as a function of time. How the motion changes with time is given by the application of Newton's IInd Law. One such particle at position \vec{r} moving with velocity \vec{v} and acted upon by a force \vec{F} is shown in figure 1. The force \vec{F} gives rise to an acceleration $\vec{a} = \frac{\vec{F}}{m}$. Notice that in general the position, the velocity and the acceleration are not in the same direction.

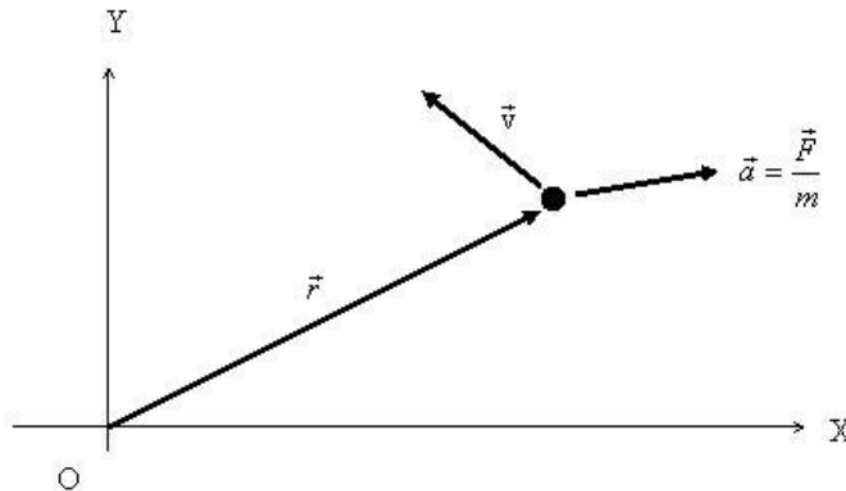


Figure 1

Each of these vectors is specified by giving its component along a set of conveniently chosen axes. For a particle moving in a plane, if we choose the Cartesian coordinate system (x-y axes) then the position is given by specifying the coordinates (x, y), velocity by its components

(v_x, v_y) and acceleration by its components $a_x = \frac{F_x}{m}$ and $a_y = \frac{F_y}{m}$. These are related by the relationship

$$v_x = \frac{dx}{dt} \quad , \quad v_y = \frac{dy}{dt}$$

and

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \quad , \quad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}$$

These expressions are easily generalized to three dimensions by including the z-component of the motion also. However, in this lecture we will be focusing on motion in a plane only. With these components the equations of motion to be solved are

$$m \frac{d^2x}{dt^2} = F_x \quad , \quad m \frac{d^2y}{dt^2} = F_y$$

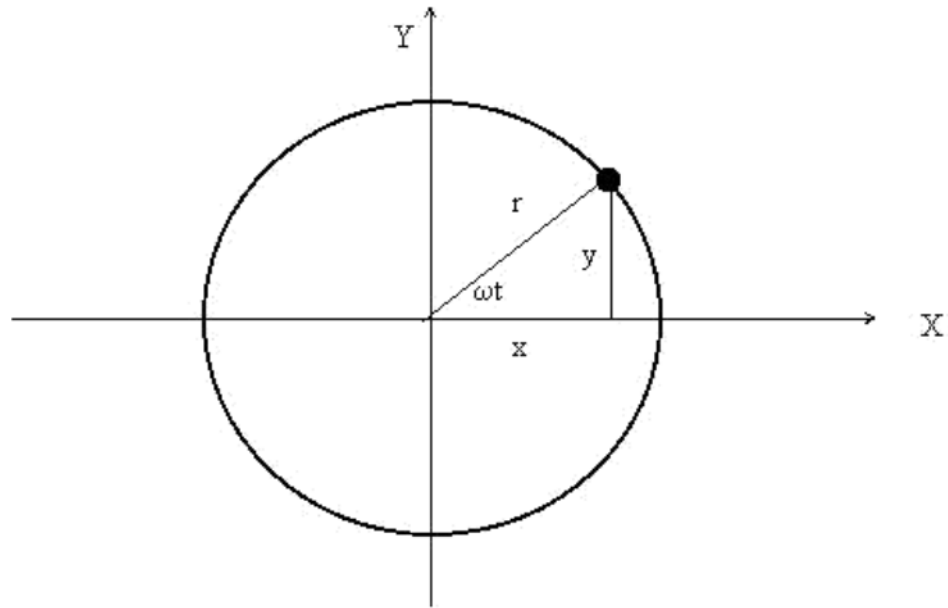
Coupled with the initial conditions solutions of these equations provide the velocity and position of a particle uniquely. However, the Cartesian system of coordinates is only one way of describing the motion of a particle. There arise many situations where describing the motion in some other coordinate system i.e., taking components along some other directions is more convenient. One such coordinate system is polar coordinates. In this lecture we discuss the use of this system to describe the motion of a particle. To introduce you to polar coordinates and how their use may make things easy, we start with the discussion of a particle in a circle.

Consider a particle is moving with a constant angular speed ω in a circle of radius R centered at the origin (see figure 2). Its x and y coordinates are given as

$$x = R \cos \omega t$$

$$y = R \sin \omega t$$

with both x and y being functions of time (see figure 2).



A particle moving with constant angular speed ω in a circle of radius R

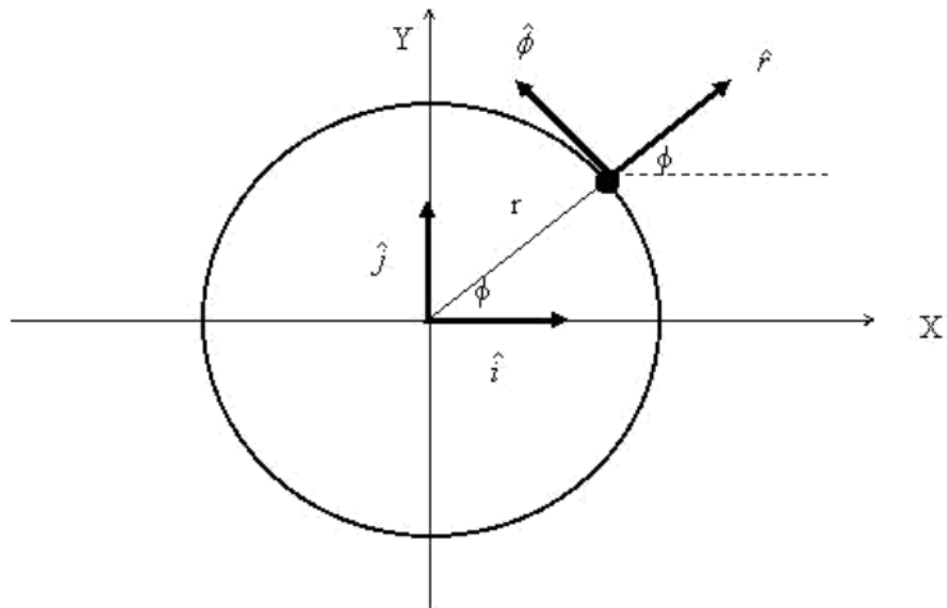
Figure 2

On the other hand, if we choose to give the position of the particle by giving its distance r from the origin and the angle Φ that the line from the origin to the particle makes with x-axis in the counter-clockwise direction, then the position is given as

$$r = R$$

$$\phi = \omega t$$

In this coordinate system, r is a constant and Φ a linear function of time. Thus there is only one variable that varies with time whereas the other one remains constant. The motion description thus is simpler. These co-ordinates (r and ϕ) are known as the planar polar coordinates. As expected, these coordinates are most useful in describing motion when there is some sort of a rotational motion. We will therefore find them useful, for example, in discussing motion of planets around the sun rotating bodies and motion of rotating objects.



Unit vectors \hat{r} and $\hat{\phi}$ in polar coordinates

Figure 3

So to start with let us set up the unit vectors in polar co-ordinates (r, Φ) . Given a point (r, Φ) , the unit vector \hat{r} is in outward radial direction and has magnitude of unity. The $\hat{\Phi}$ unit vector is also of magnitude unity and is perpendicular to \hat{r} and in the direction of increasing Φ (see figure

3). Obviously the dot product $\hat{r} \cdot \hat{\phi} = 0$. In terms of the unit vectors in x and y direction these are given as

$$\begin{aligned}\hat{r} &= \cos \phi \hat{i} + \sin \phi \hat{j} \\ \hat{\phi} &= -\sin \phi \hat{i} + \cos \phi \hat{j}\end{aligned}$$

As is clear from these expressions the direction of \hat{r} and $\hat{\phi}$ is not fixed but depends on the angle Φ . On the other hand, it does not depend on r . If we go along a radius, \hat{r} and $\hat{\phi}$ remain unchanged as we move (recall that two parallel vectors of same magnitude are equal). But that is not the case if Φ is changed.

The position of a particle in polar co-ordinates is given by writing

$$\vec{r} = r \hat{r}$$

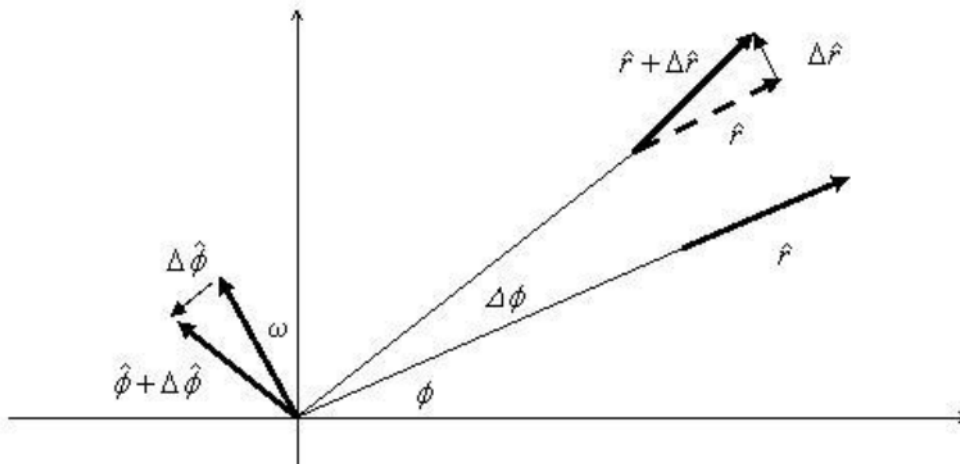
As the particle moves about, \vec{r} changes. Does this mean that the velocity

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt} \hat{r} ?$$

The answer is no. As already discussed \hat{r} is a function of Φ , the angle from the x-axis. Thus as a particle moves such that the angle Φ changes with time, the unit vector \hat{r} also changes. Its derivative with respect to time is therefore not zero. Thus the correct expression for \vec{v} is

$$\vec{v} = \left(\frac{dr}{dt} \right) \hat{r} + r \left(\frac{d\hat{r}}{dt} \right)$$

Let us now calculate $\left(\frac{d\hat{r}}{dt} \right)$. As already stated, \hat{r} does not change as one moves radially in or out. Thus \hat{r} changes only if Φ changes. Let us now calculate this change (figure 4)



Change in unit vector \hat{r} and $\hat{\phi}$ as angle ϕ is changed to $\phi + \Delta\phi$

Figure 4

As is clear from the figure

$$\begin{aligned} \Delta \hat{r} &= \Delta \phi \hat{\phi} \\ \Rightarrow \frac{d\hat{r}}{dt} &= \left(\frac{\Delta \phi}{\Delta t} \right) \hat{\phi} = \dot{\phi} \hat{\phi} \end{aligned}$$

where the dot on top of a quantity denotes its time derivative. The expression above can also be derived mathematically as follows:

$$\begin{aligned}
 \frac{d\hat{r}}{dt} &= \frac{d}{dt} (\cos \phi \hat{i} + \sin \phi \hat{j}) \\
 &= \dot{\phi} (-\sin \phi \hat{i} + \cos \phi \hat{j}) \\
 &= \dot{\phi} \hat{\phi}
 \end{aligned}$$

Thus the velocity of a particle is given as

$$\vec{v} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$$

We note that the unit vectors in polar coordinates keep changing as the particle moves because they are given by the particles current position. Thus even if a particle were moving with a constant velocity, the components of velocity along the radial and the directions will change. Let us calculate the velocity of a particle moving in a circle with a constant angular speed. For such a particle

$$\dot{r} = 0 \quad \text{and} \quad \dot{\phi} = \omega$$

so the velocity is given as

$$\vec{v} = R\omega\hat{\phi}$$

This is a well known result: the velocity of a particle moving in a circle with a constant angular speed is in the tangential direction and its magnitude is $R\omega$. How about the acceleration in polar coordinates? This is the derivative of \vec{v} with respect to time. Thus

$$\vec{a} = \left(\frac{d\vec{v}}{dt} \right) = \left(\frac{d}{dt} \right) (\dot{r}\hat{r} + r\dot{\phi}\hat{\phi})$$

As was the case with the unit vector \hat{r} , the unit vector $\hat{\phi}$ also is a function of the polar angle Φ and as such changes as the particle moves about. Thus in calculating the acceleration, time derivative of $\hat{\phi}$ also should be taken into account. From figure 4 it is clear that

$$\begin{aligned}
 \Delta \hat{\phi} &= -\Delta \phi \hat{r} \\
 \Rightarrow \frac{d\hat{\phi}}{dt} &= -\left(\frac{\Delta \phi}{\Delta t} \right) \hat{r} = -\dot{\phi} \hat{r}
 \end{aligned}$$

This can also be derived mathematically as

$$\begin{aligned}
 \frac{d\hat{\phi}}{dt} &= \frac{d}{dt}(-\sin \phi \hat{i} + \cos \phi \hat{j}) \\
 &= \dot{\phi}(-\cos \phi \hat{i} - \sin \phi \hat{j}) \\
 &= -\dot{\phi} \hat{r}
 \end{aligned}$$

Using this derivative and the chain rule for differentiation, we get

$$\begin{aligned}
 \vec{a} &= \left(\frac{d}{dt} \right) (\dot{r} \hat{r} + r \dot{\phi} \hat{\phi}) \\
 &= \ddot{r} \hat{r} + \dot{r} \frac{d\hat{r}}{dt} + \dot{r} \dot{\phi} \hat{\phi} + r \ddot{\phi} \hat{\phi} + r \dot{\phi} \frac{d\hat{\phi}}{dt} \\
 &= (\ddot{r} - r \dot{\phi}^2) \hat{r} + (r \ddot{\phi} + 2\dot{r} \dot{\phi}) \hat{\phi}
 \end{aligned}$$

You can see that the expression is a little complicated. The complexity of the expression arises because the unit vectors are changing as the particle moves. You can check for yourself that for a particle moving with a constant velocity, the expression above will give zero acceleration. Despite little complicated expressions for the acceleration, employing polar coordinates becomes really useful in situations where motion is circular-like as we will see in two standard examples later. Let us first go to one familiar example of a particle moving in a circle for which $r = R$,

$\dot{\phi} = \omega$. This gives

$$\vec{a} = -R\omega^2 \hat{r}$$

which is the correct answer for the centripetal acceleration. For this reason $r\dot{\phi}^2$ is known as the centripetal term. Let us now solve an example of mechanics using polar co-ordinates.

Example 1: A bead of mass m can slide without friction on a straight thin wire moving with constant angular speed ω in a horizontal plane (figure 5). If we leave the bead with zero initial radial velocity at $r = R$, we wish to describe its subsequent motion and also find the horizontal force applied by the wire on the bead.

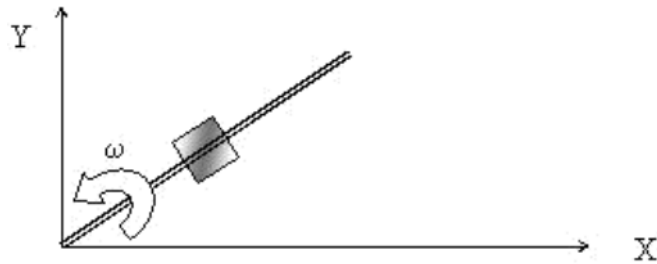


Figure 5

To see the usefulness of polar coordinates, try to write equations of motion for the bead in the Cartesian coordinates. This I leave for you to do. We solve the problem using polar co-ordinates. Thus at any instant the acceleration is given by the formula

$$\vec{a} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi}$$

We emphasize that the expression above gives the components of the acceleration along the radial and the ϕ directions which are not fixed in space but are changing continuously. It is given that $\dot{\phi} = \omega$ (a constant) which also means that $\ddot{\phi} = 0$. The acceleration of the bead on the wire is therefore

$$\vec{a} = (\ddot{r} - r\omega^2)\hat{r} + 2\dot{r}\omega\hat{\phi}$$

Since there is no friction, the wire does not apply any radial force on the bead. Therefore

$$\ddot{r} - r\omega^2 = 0$$

You can check by substitution that the solution for the equation above is

$$r = Ae^{\omega t} + Be^{-\omega t}$$

where A and B are two constants to be determined from the initial conditions. Differentiating the equation above gives

$$\dot{r} = \omega(Ae^{\omega t} - Be^{-\omega t})$$

Thus acceleration perpendicular to wire is

$$a_{\phi} = 2\dot{r}\omega$$

$$= 2\omega^2(Ae^{\omega t} - Be^{-\omega t})$$

So the horizontal force applied by wire is

$$F_{\phi} = ma_{\phi} = 2m\omega^2(Ae^{\omega t} - Be^{-\omega t})$$

Of course because the unit vectors employed change direction continuously, the force above is also in different directions at different times with the magnitude given by the expression above. To determinate A and B, we substitute $t = 0$ in the expressions derived for the radius and the radial speed and equate them to their vales given at that time. This gives

$$A + B = R$$

$$A - B = 0$$

$$\Rightarrow A = B = \frac{R}{2}$$

This leads to the answer

$$r = \frac{R}{2}(e^{\omega t} - e^{-\omega t})$$

$$\dot{r} = \frac{R\omega}{2}(e^{\omega t} - e^{-\omega t})$$

$$F = m\omega^2 R(e^{\omega t} - e^{-\omega t})$$

Example 2: A particle, tied to a string, is moving on a smooth frictionless table in a circle of radius r_0 with an angular speed ω_0 . The string is pulled in slowly through a hole in the middle of the table with constant speed V . We want to find the change in its speed as a function of time and also the force required for the string to be pulled (figure 6).

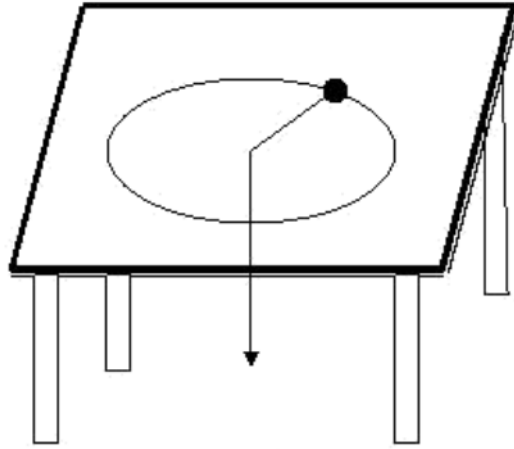


Figure 6

The mass, when pulled in, is moving under the influence of an inwardly directed radial force $-F\hat{r}$. Although the force keeps changing its direction depending upon where the particle is, it always remains radial. The expression for the acceleration of the particle in the polar coordinates is

$$\vec{a} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi}$$

Since it is given $\dot{r} = -V$, which means $\ddot{r} = 0$, and the force is only in $-\hat{r}$ direction, we have

$$\begin{aligned}\ddot{r} - r\dot{\phi}^2 &= -r\dot{\phi}^2 = -\frac{F}{m} \\ \text{or } mr\dot{\phi}^2 &= F\end{aligned}$$

Since there is no force component in the Φ direction, we have

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0$$

Multiply both sides of this equation by r to get

$$\begin{aligned}r^2\ddot{\phi} + 2r\dot{r}\dot{\phi} &= 0 \\ \text{or } \frac{d}{dt}(r^2\dot{\phi}) &= 0 \\ \text{or } r^2\dot{\phi} &= \text{constant}\end{aligned}$$

Since $r = r_0 - Vt$, the equation above gives

$$r_o^2 \omega_o = (r_o - Vt)^2 \omega$$

$$\text{or } \omega = r_o^2 \omega_o / (r_o - Vt)^2$$

The force $F = mr\dot{\phi}^2$ pulling the string in is therefore

$$F = m \frac{r_o^4 \omega_o}{(r_o - Vt)^3}$$

In solving this example, we see that for forces in radial direction $r^2 \omega = \text{constant}$, which is nothing by a statement of the conservation of angular momentum. We will discuss it more later when we study angular momentum.

After introducing the planar polar coordinates, we now briefly describe what are the other coordinate systems in three dimensions. A natural extension of planar coordinates is the cylindrical coordinate system. This arises when we add the third-z direction to planar polar coordinates. See figure 7.

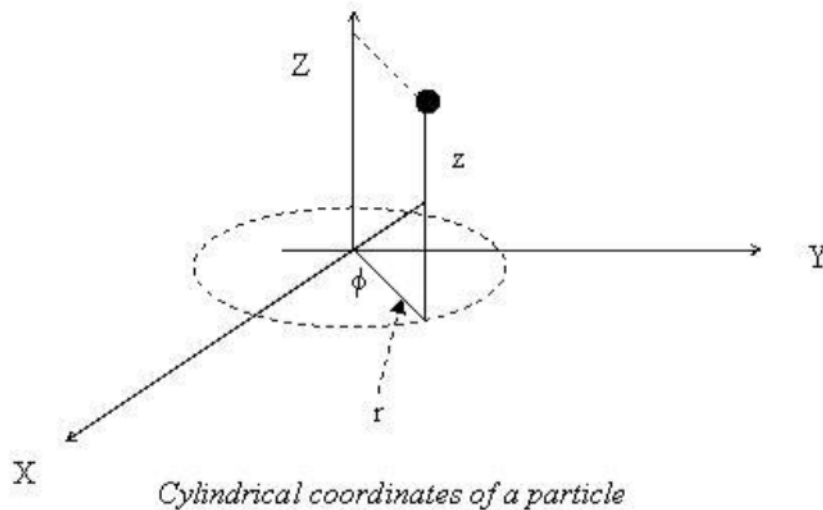


Figure 7

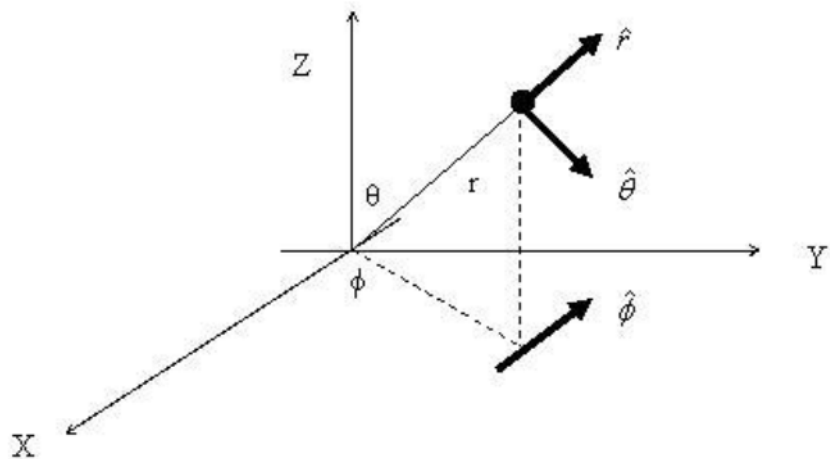
The position of a particle is described by (r, ϕ, z) with the corresponding unit vectors being $(\hat{r}, \hat{\phi}$ and $\hat{z})$. In this case the \hat{z} unit vector is a constant and $(\hat{r}, \hat{\phi})$ are given as in the planar polar co-ordinates so that

$$\begin{aligned}\hat{r} &= \cos \phi \hat{i} + \sin \phi \hat{j} \\ \hat{\phi} &= -\sin \phi \hat{i} + \cos \phi \hat{j} \\ \hat{z} &= \hat{k}\end{aligned}$$

Thus the expressions for all the quantities are similar to those for planar polar co-ordinates except that \hat{z} direction is also added. As a result,

$$\begin{aligned}\vec{r} &= r\hat{r} + z\hat{z} \\ \vec{v} &= \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} + \dot{z}\hat{z} \\ \vec{a} &= (\ddot{r} - r\dot{\phi}^2)\hat{r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi} + \ddot{z}\hat{z}\end{aligned}$$

We now introduce another set of coordinates, the spherical polar coordinates, in three dimensions. A point in these coordinates is specifically by its distance from the centre r , the angle θ that the line joining the point to origin makes with the z-axis and the angle ϕ that the projection of this line on the (xy) plane makes with the x-axis. Thus a point is specified by (r, θ, ϕ) (see figure 8).



Spherical coordinates of a point and the unit vectors \hat{r} , $\hat{\theta}$ and $\hat{\phi}$

Figure 8

Thus (x, y, z) co-ordinates for a point (r, θ, ϕ) are

$$\begin{aligned}x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta\end{aligned}$$

The unit vectors are given as $\hat{r}, \hat{\theta}$ and $\hat{\phi}$ with

$$\hat{r} = \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}$$

$\hat{\theta}$ unit vector points in a direction below the (xy) plane making an angle θ from the (xy) plane. So it is given as

$$\hat{\theta} = \cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k}$$

And $\hat{\phi}$ is in the (xy) plane and is given as

$$\hat{\phi} = -\sin \phi \hat{i} + \cos \phi \hat{j}$$

which is the same as for planar polar coordinates. As is clear, the unit vectors in this case are also position dependent and change as the particle position changes. This affects the expression for velocities and acceleration when they are expressed in spherical coordinates.

Let us evaluate the time derivatives of $\hat{r}, \hat{\theta}$ and $\hat{\phi}$ geometrically. The unit vector \hat{r} does not depend on r but changes with θ . This gives

$$\Delta \hat{r}(\theta \rightarrow \theta + \Delta \theta) = \Delta \theta \hat{\theta}$$

Similarly when θ is fixed and ϕ changes, we get

$$\Delta \hat{r} = (\phi \rightarrow \phi + \Delta \phi) = \sin \theta \Delta \phi \hat{\phi}$$

When we combine the two results we get

$$\Delta \hat{r} = \Delta \theta \hat{\theta} + \sin \theta \Delta \phi \hat{\phi}$$

which gives

$$\dot{\hat{r}} = \dot{\theta} \hat{\theta} + \sin \theta \dot{\phi} \hat{\phi}$$

Thus the expression for velocity in spherical coordinates is

$$\begin{aligned} \vec{v} = \dot{\vec{r}} &= \frac{d}{dt} (r \hat{r}) \\ &= \dot{r} \hat{r} + r \dot{\theta} \hat{\theta} + r \sin \theta \dot{\phi} \hat{\phi} \end{aligned}$$

We leave the calculation of $\dot{\hat{\theta}}$ and $\dot{\hat{\phi}}$ and the acceleration as an exercise. We end this brief introduction to spherical coordinates by noting that spherical polar coordinates can be those of as two plane polar coordinates systems : one the plane of radius vector and the z-axis with (r, θ) as planar coordinates and the other the (xy) plane with $(r \sin \theta, \phi)$ as the planar polar coordinates.

UNIT – II

KINETICS OF PARTICLE

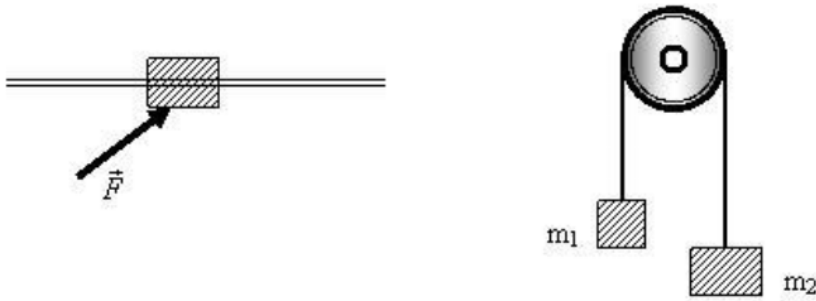
Motion with constraints

In this lecture we are going to deal with motion of particles when they move under constraints applied on their motion. Of course the motion is determined by Newton's second law i.e., by solving the equation of motion

$$m\vec{a} = m \frac{d^2\vec{r}}{dt^2} = \vec{F}$$

where \vec{F} is the total force – which is the sum of the externally applied and those arising from other particles as well as the constraints in the system - acting on a body of mass m and is

producing an acceleration $\vec{a} = \frac{d^2\vec{r}}{dt^2}$. Recall from *lecture 9* that constraints are the restrictions applied on the movement of a body by various means and are brought about by *constraint forces*. For example, I may restrict the body to move along a straight wire (see figure 1). In that case the component of \vec{F} only along the wire will affect the motion of the mass (if there is no friction) and its perpendicular component will be nullified by the normal reaction of the wire, which is the constraint force in this case. As another common example of constrained motion take the motion of two masses at the end of a rope going over a frictionless pulley (Atwood's machine) also shown in figure 1.



Two examples of constrained motion

Figure 1

In this case also, the motion of one mass is determined by not only by the gravitational force on it alone but also by the weight of the other mass. Thus the two masses are not fully free to move under their own weight and the motion is constrained. The constrained is brought about through tension in the rope, which is then the constraint force.

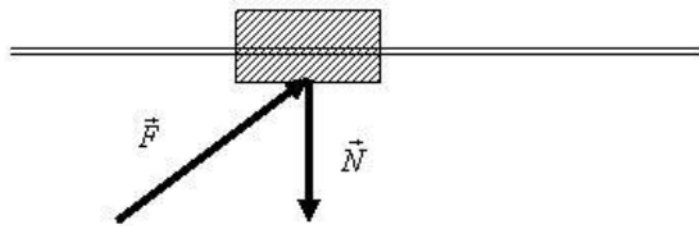
We have seen two simple examples of constrained motion. We make an observation that constraints can be caused either by restricting the motion externally, as was the case for a mass on a wire, or by the presence of other bodies that are themselves moving, as in the example of two masses over a pulley. In lecture 9 we had introduced these concepts and stopped at that. However, for obtaining the positions and velocities of particles under constraints, we wish to express these constraints mathematically and account for them while solving the equations of motion. This is what this lecture is going to be about.

Let us start with the example of a mass on a straight wire (say in x direction). The constraint that the mass moves only in the x-direction is equivalent to saying that

$$y = \text{constant}$$

$$z = \text{constant}$$

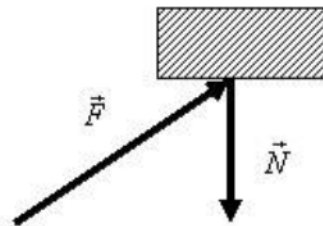
This is how we mathematically express the constraint that the mass moves only along the x-axis. As pointed out earlier, to keep the y and the z coordinates of the mass unchanged, the wire applied a normal force on the mass to cancel the perpendicular (to the wire) component of the applied force so that the net force is along the wire. This normal reaction is the constraint force (figure 2). Notice that all that the wire does to the mass, as far as its motion is concerned, is represented by this force.



A mass constrained to move on a wire under applied force \vec{F} . Normal force \vec{N} is the constraint force. Note that the sum of these two forces is along the wire.

Figure 2

To study the motion of the mass all I need to look at are only the forces – external and constraint forces - acting on the mass. In this case the wire is represented by the normal force that it applies. Recall from *lecture 4* that such a diagram is called a *free-body diagram* . The advantage of drawing a free-body diagram is that it identifies the relevant quantities to write the equation of motion. In the present case the free-body diagram of the mass is given in figure 3.



Free-body diagram of a mass on a wire

Figure 3

Let us now write the equations of motion for the body in terms of its x , y and z -components :

$$\begin{aligned} m\ddot{x} &= F_x \\ m\ddot{y} &= F_y + N_y \\ m\ddot{z} &= F_z + N_z \end{aligned}$$

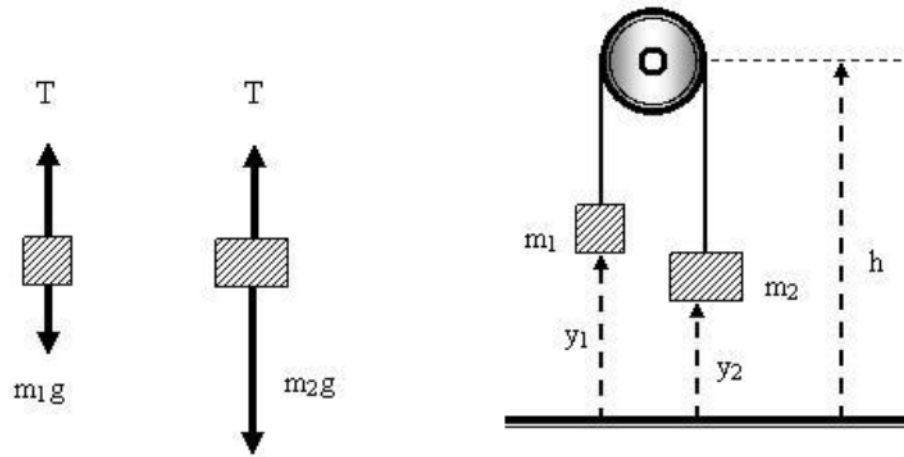
Let us count how many unknown are there? The unknowns are x , y , z , N_y , and N_z , (numbering five \vec{F} is given). But there are only three equations. How do we find the other two equations? For this recall that the two of the unknowns, N_x and N_y , arise because of the constraints. And it is these constraints that provide the two more equations needed for a solution. The constraints that $y = \text{constant}$ and $z = \text{constant}$ imply that

$$\begin{aligned} \ddot{y} &= 0 \\ \ddot{z} &= 0 \end{aligned}$$

With these two additional equations, we now have five equations and five unknowns. Thus and we can solve for x , y , z and N_x and N_y in terms of given parameters of the problem.

Let us now look at the other problem of two masses hanging on the sides of a frictionless pulley (see figure 1), a special case of Atwood's Machine. For simplicity we take the pulley and the rope to be massless. Let the masses be m_1 & m_2 . In this problem also the motion is in only one direction the vertical direction so we are going to ignore the other two dimensions. In this problem the constraint is that the two masses move together and it is effected by the rope. As

noted above, the force of constraint therefore is the tension T in the rope. Let us now make their free-body diagrams for the two moving masses m_1 and m_2 . We measure all distances from the ground and let the distance of m_1 be y_1 and that of m_2 be y_2 . Please see figure 4.



Free-body diagrams of m_1 and m_2 and their distances y_1 and y_2 from the ground. The pulley is at height h .

Figure 4

Equation of motion for m_1 and m_2 are

$$\begin{aligned} m_1 \ddot{y}_1 &= T - m_1 g \\ m_2 \ddot{y}_2 &= T - m_2 g \end{aligned}$$

The tension T is the same on both sides because rope and pulley both are massless and the pulley is also frictionless. These are two equations and there are three unknowns: y_1, y_2 and T . The tension T arises because of constraint so the constraint itself provides the desired third equation. In this case the constraint is that the length of the rope is constant. This can be expressed mathematically as (see figure 4 for meaning of symbols)

$$(h - y_1) + (h - y_2) = \text{length of rope} - \pi R = \text{constant}$$

where R is the radius of the pulley. Differentiating this equation twice with respect to time gives

$$\ddot{y}_1 + \ddot{y}_2 = 0 \quad \text{or} \quad \ddot{y}_1 = -\ddot{y}_2$$

We now have three equations for three unknowns:

$$\begin{aligned}
m_1 \ddot{y}_1 &= T - m_1 g \\
m_2 \ddot{y}_2 &= T - m_2 g \\
\ddot{y}_1 &= -\ddot{y}_2
\end{aligned}$$

Solving these equations gives

$$\ddot{y}_1 = \left(\frac{m_2 - m_1}{m_1 + m_2} \right) g \quad \text{and} \quad \ddot{y}_2 = -\left(\frac{m_2 - m_1}{m_1 + m_2} \right) g$$

a result that you already know. Thus if $m_2 > m_1$, m_1 accelerates up.

Through these two simple examples, I have identified sequential steps that we take in solving a problem involving constraints I now summarize these steps:

1. Identify the constraints and forces of constraints in the given problem;
2. Make free body diagrams of different bodies taking part in the motion. Let me remind you in making free body diagram take the body and show all the forces - applied and those of constraints - on the body;
3. Write equations of motion for each subsystem/body. At this stage the number of equations will be less than the number of variables in the problem;
4. Write the constraint equations. They will provide the missing equations (This happens because each constraint introduces a constraint force which becomes the additional unknown);
5. Solve the equations.

Let us now apply the procedure outlined above to slightly more difficult examples.

Example 1: There are three massless and frictionless pulleys P1, P2 and P3. P1 and P2 are fixed and P3 can move up and down, as shown in figure 5. A massless rope R1 passes over the pulleys as shown and two masses m_1 and m_2 attached at its ends. A third mass m_3 is hanging from P3 by a rope R2 of fixed length. Find the acceleration of the three masses.

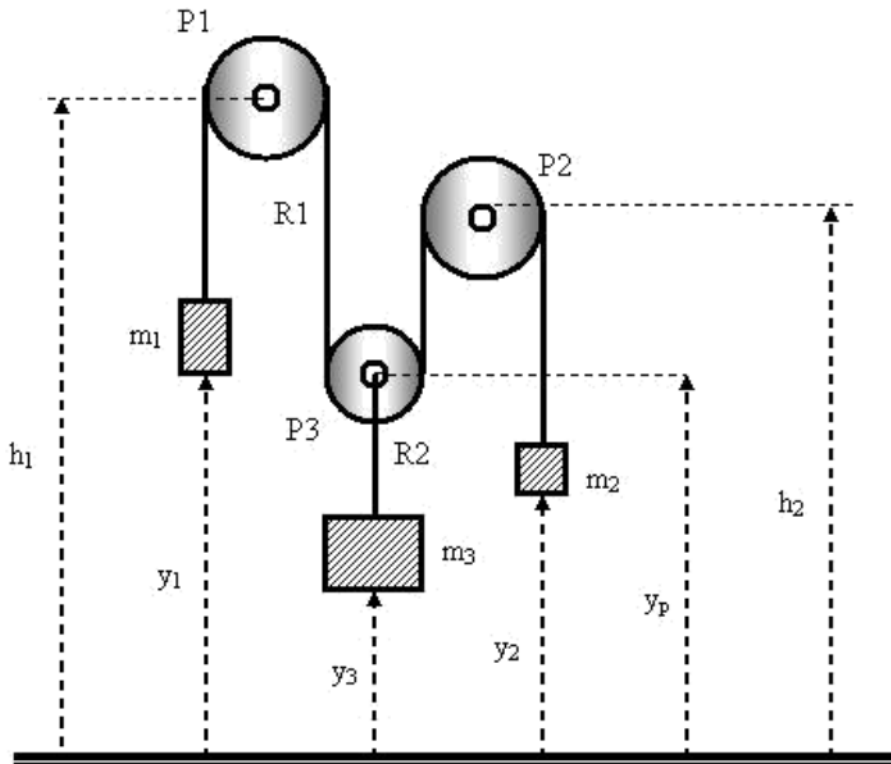
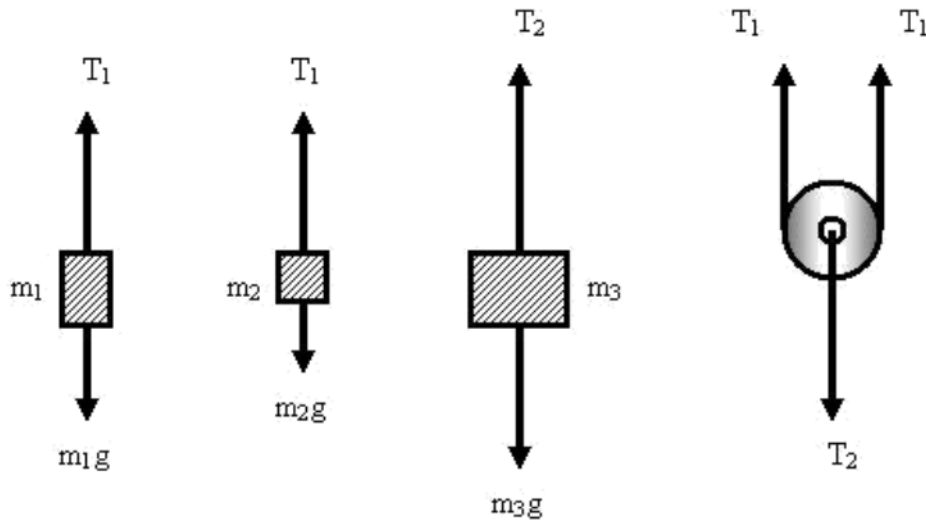


Figure 5

In figure 5 we have also shown the distances of different pulleys and masses from the ground, with the vertically up direction taken to be positive. The heights h_1 and h_2 of pulleys P1 and P2, respectively, are fixed whereas height y_p of pulley P3 can change. We go about solving the problem according to the steps given above.

Step 1: We identify two constraints and the forces of constraints as: rope R1 has fixed length with the force of constraint being tension T_1 in the rope. The other constraint is that rope R2 has fixed length with the tension T_2 in the rope as the constraint force. Because of massless pulleys and ropes and frictionless surfaces T_1 is the same throughout rope R1.

Step 2 : Make free-body diagrams of the subsystems. We consider only those subsystems that can move. Thus we make free-body diagram of each mass and the pulley P3 as shown in figure 6.



Free-body diagram for the three masses and pulley P3

Figure 6

Step 3 : By looking at the free-body diagrams, write equations of motion for each subsystem. In terms of the distances shown in figure 5, we get

$$\begin{aligned} m_1 \ddot{y}_1 &= T_1 - m_1 g \\ m_2 \ddot{y}_2 &= T_1 - m_2 g \\ m_3 \ddot{y}_3 &= T_2 - m_3 g \end{aligned}$$

and because the pulley is massless

$$T_2 - 2T_1 = 0$$

Thus equations of motion give four equations. However there are six unknowns viz.

y_1, y_2, y_3, T_1, T_2 , and y_p . Their number exceeds the number of equations obtained so far by two.

Step 4 : The additional two equations are provided by the constraint equations. The constraint that rope R1 is of fixed length is expressed as (see figure 5 for the variables used)

$$\begin{aligned} (h_1 - y_1) + (h_1 + h_2 - 2y_p) + (h_2 - y_2) &= \text{constant} \\ \text{or } y_1 + y_2 + 2y_p &= \text{constant} \end{aligned}$$

Differentiating this equation twice with respect to time gives

$$\ddot{y}_1 + \ddot{y}_2 + 2\ddot{y}_p = 0$$

The second constraint that rope R2 is of fixed length is equivalent to

$$y_p - y_3 = \text{constant}$$

which upon differentiating gives

$$\ddot{y}_p - \ddot{y}_3 = 0$$

Thus the equations that describe the motion of the system fully are:

$$\begin{array}{ll} m_1 \ddot{y}_1 = T_1 - m_1 g & T_2 - 2T_1 = 0 \\ m_2 \ddot{y}_2 = T_1 - m_2 g & \ddot{y}_1 + \ddot{y}_2 + 2\ddot{y}_p = 0 \\ m_3 \ddot{y}_3 = T_2 - m_3 g & \ddot{y}_p - \ddot{y}_3 = 0 \end{array}$$

I will leave Step 5 – that is solving the equations – for you to do but give you partial answer. It is

$$\ddot{y}_3 = \frac{4m_1 m_2 - (m_1 + m_2)m_3}{4m_1 m_2 + (m_1 + m_2)m_3} g$$

I would now like you to try a similar problem but with slight difference. Let us attach the centre of the third pulley to a spring of spring constant k (see figure 7). Then find the equations of motion for the two masses and solve them.

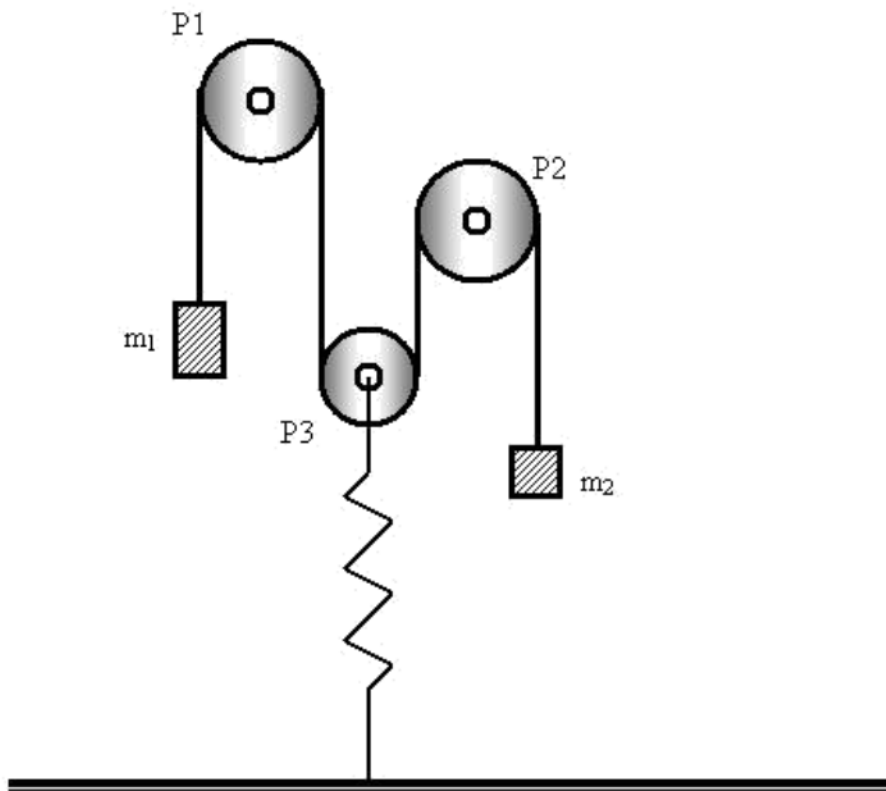


Figure 7

Example 2 :As another example of constrained motion we take a small block of mass m sliding down on a cylindrical surface from its top (figure 8). The question we ask is at what angle from the horizontal would the mass slide off the surface of the cylinder.

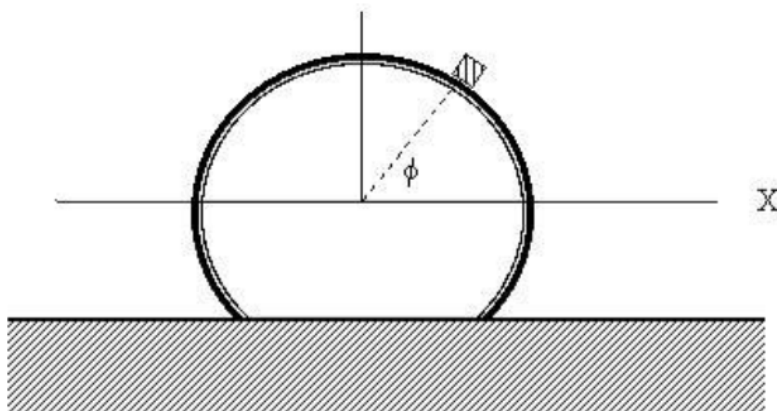
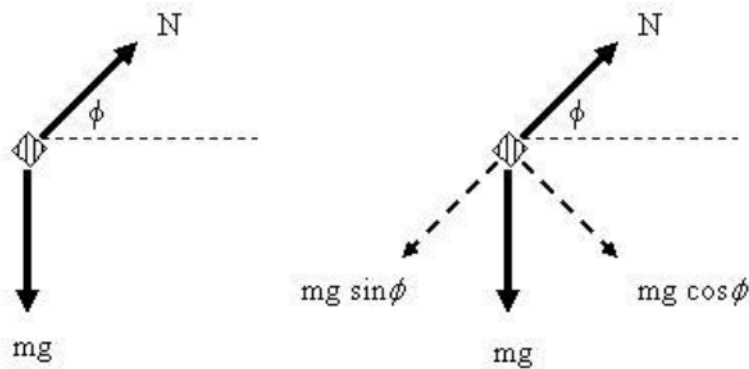


Figure 8

Since this problem involves motion along a circular path I would use planar polar coordinates. I take the origin at the centre of the cylinder and let the x-axis be along the horizontal and y-axis along the vertical. Assume that the radius of the cylinder is R . The constraint in this problem is that $r = \text{constant} = R$. The corresponding constraint force is the normal reaction N of the cylindrical surface on the block. The free-body diagram of the mass on the cylinder is shown in figure 9.



Free-body diagram of a mass on a cylinder (left) and the components of the weight mg in the radial and the ϕ directions (right)

Figure 9

We now write the equations of motion in the planar polar coordinates. That gives in the \hat{r} direction

$$m(\ddot{r} - r\dot{\phi}^2) = N - mg \sin \phi$$

and in the $\hat{\phi}$ direction

$$m(r\ddot{\phi} - 2\dot{r}\dot{\phi}) = -mg \cos \phi$$

We again have three variables (r , ϕ and N) but only two equations. The third equation is provided by the equation of constraint i.e.

$$r = \text{constant} = R$$

which gives

$$\dot{r} = \ddot{r} = 0$$

With this the equations to be solved are

$$-mR\dot{\phi}^2 = N - mg \sin \phi \quad mR\ddot{\phi} = -mg \cos \phi$$

To solve these we use

$$\begin{aligned}\ddot{\phi} &= \frac{d}{dt}(\dot{\phi}) = \frac{d}{d\phi}(\dot{\phi}) \cdot \frac{d\phi}{dt} \\ &= \dot{\phi} \cdot \frac{d}{d\phi}(\dot{\phi}) \\ &= \frac{1}{2} \frac{d}{d\phi}(\dot{\phi}^2)\end{aligned}$$

Substituting this in the equation for $\ddot{\phi}$ above gives

$$\begin{aligned}\frac{mR}{2} \frac{d}{d\phi}(\dot{\phi}^2) &= -mg \cos \phi \\ \Rightarrow R\dot{\phi}^2 &= -2g \sin \phi \Big|_{\pi/2}^{\phi} \\ &= 2g(1 - \sin \phi)\end{aligned}$$

This when substituted in the equation for $R\dot{\phi}^2$ leads to

$$3mg \sin \phi = N + 2mg$$

The point when the mass slips off the cylinder is where N becomes zero. So the corresponding ϕ is given by

$$3mg \sin \phi = 2mg \quad \text{or} \quad \sin \phi = \frac{2}{3}$$

Example 3: Let us take one more example of constrained motion when two bodies are involved. I put a block of mass m on a wedge of mass M with wedge angle θ (see figure 10). The wedge is free to move on a frictionless plane. There is no friction also between m and M . We wish to find the resulting motion.

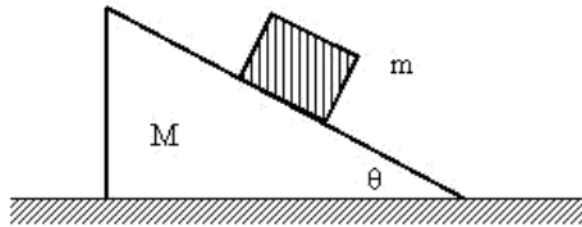


Figure 10

There are clearly two subsystems, the masses m and M . There are two constraints in the system. Constraint one is that the mass m moves along the edge of the wedge so its x and y components are not independent. The other constraint is that the wedge moves only in the x direction. The constraint forces are obviously the normal reaction N_1 on mass m by the wedge and the normal reaction N_2 on the wedge by the ground. The free-body diagrams for the two subsystems are as shown in figure 11.

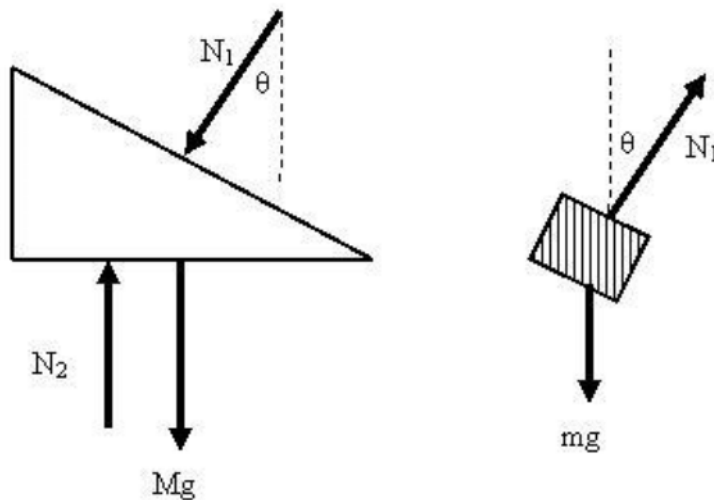


Figure 11

Notice that in the free-body diagram of the wedge, there is no mg of block. It is all accounted for by N_1 . To set up the equations of motion, let us choose our co-ordinates system as follows (see figure 12): Let the coordinate of the right-hand side lower corner of the wedge be given the co-ordinates (x_1, y_1) and let the co-ordinates of the block be (x_2, y_2) .

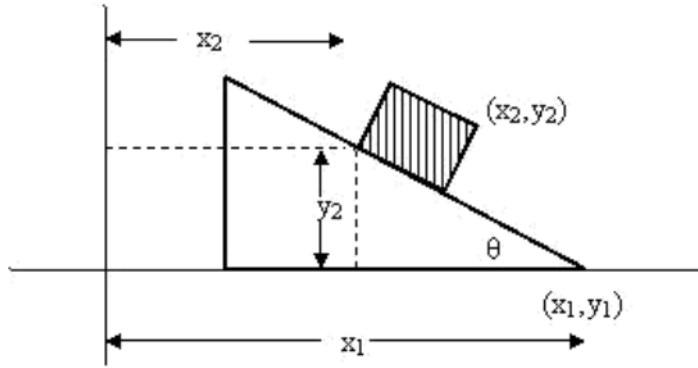


Figure 12

The equations of motion in terms of these coordinates are:

$$\begin{aligned} m\ddot{x}_2 &= N_1 \sin \theta & M\ddot{x}_1 &= -N_1 \sin \theta \\ m\ddot{y}_2 &= N_1 \cos \theta - mg & M\ddot{y}_1 &= -N_1 \cos \theta - mg + N_2 \end{aligned}$$

For the six variables - $(x_1, y_1), (x_2, y_2), N_1$ and N_2 - of the system, we need two more equations, which are provided by the constraints equations. These are

$$y_1 = 0 \text{ which gives } \dot{y}_1 = \ddot{y}_1 = 0$$

and

$$\frac{y_2}{x_1 - x_2} = \text{constant} = \tan \theta$$

which gives

$$\dot{y}_2 = (\dot{x}_1 - \dot{x}_2) \tan \theta \quad \text{and} \quad \ddot{y}_2 = (\ddot{x}_1 - \ddot{x}_2) \tan \theta$$

Thus the equations to be solved are

$$\begin{aligned} m\ddot{x}_2 &= N_1 \sin \theta & M\ddot{y}_1 &= -N_1 \cos \theta - mg + N_2 \\ m\ddot{y}_2 &= N_1 \cos \theta - mg & \ddot{y}_1 &= 0 \\ M\ddot{x}_1 &= -N_1 \sin \theta & \ddot{y}_2 &= (\ddot{x}_1 - \ddot{x}_2) \tan \theta \end{aligned}$$

These equations can now be solved to get all the variables as a function of time. That task is left for you. I'll leave you with answers for N_1 :

$$N_1 = \frac{mg \cos \theta}{\left(1 + \frac{m}{M} \sin^2 \theta\right)}$$

Motion with friction and drag

We have been looking at the constrained motion of particles and found that in solving the problems we make free-body diagrams and look at the motion of each subsystem independently. Then the motion of individual subsystem is linked through constraints that they impose on each other. The example that we took were Atwood's machine and a mass sliding on a wedge. However, in these examples we neglected a ubiquitous force which is the force of friction. In this lecture we take this into account and solve problems involving the friction

We would take into account two kinds of frictional forces - one that arises when two solid bodies are in contact and the other that arises when a body is moving through a liquid, the viscous force. Let us first consider the case when two solid bodies are moving against each other. A detailed discussion about the nature of frictional force and its relationship with the normal reaction has already been presented in lecture 6. We start with a review of the main points discussed there.

If there is a tendency between two bodies to slide against each other, or if one body is sliding over a surface, the friction between the two bodies resists this motion. Question is whether this is a constant force or adjusts itself. It is experimentally observed that the maximum frictional force

f_{\max} that a surface can apply on an object is

$$f_{\max} = \mu N$$

where N is the normal reaction of the surface on the body and μ is the coefficient of friction; its value is different for the static and dynamic case. Thus there are two coefficients of friction between two surfaces: static coefficient of static friction μ_s and the coefficient of dynamic friction μ_k , with the latter being smaller than the former. Further, μ_s is always observed to be less than 1. And the direction of frictional force is such that it opposes the motion or the tendency to move.

Let us now take a couple of standard examples involving friction similar to those solved in lecture 6.

Example 1: We put a block of 5kg on top a 10kg block. They are then attached through a massless and frictionless pulley to a mass M as shown in figure 1. The coefficient of friction between all surfaces for both static and dynamic friction is 0.5. What is the acceleration for (a) $M = 20\text{kg}$ and (b) $M = 40\text{kg}$ ($g = 9.8\text{m/s}^2$)?

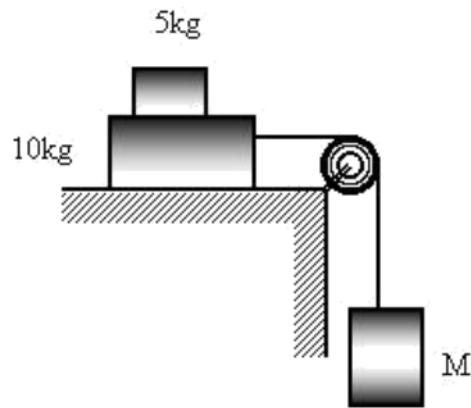
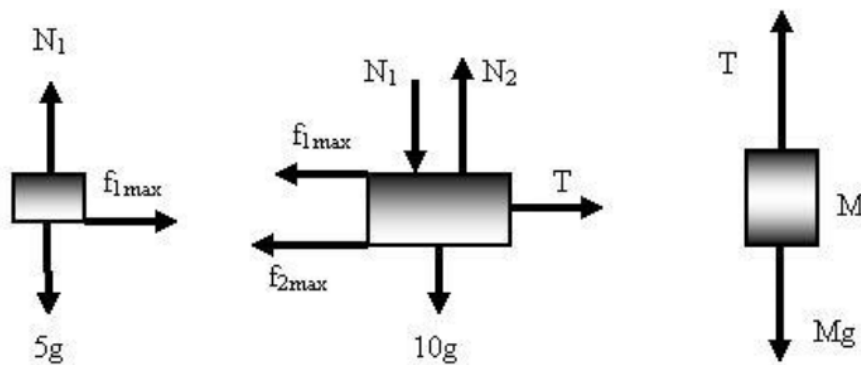


Figure 1

What we should see in solution of this problem is the maximum possible acceleration that the 5kg block can have, and then solve for the mass M_0 that will give this acceleration for both the 5kg and the 10kg blocks. If M is less than M_0 , both the blocks will move together. On the other hand, if M exceeds M_0 , the blocks will slip on each other.

To start the calculations I show in figure 2 the free body diagrams of all the masses with maximum possible friction



Free body diagrams of all the blocks in figure with maximum possible friction.

Figure 2

Looking at the 5kg block, we see that

$$N_1 = 5g \quad \Rightarrow \quad f_{1\max} = 0.5 \times 5 \times 9.8 = 24.5N$$

The maximum possible acceleration for the 5kg mass is

$$a_{1\max} = \mu g = 4.9ms^{-2}$$

Let me now calculate M_0 corresponding to this acceleration. The corresponding equations for the 10kg block are

$$\begin{aligned} N_2 &= N_1 + 10g = 147N & \Rightarrow & \quad f_{2\max} = 0.5 \times 147 = 73.5N \\ T - f_{1\max} - f_{2\max} &= 10a_{1\max} & \Rightarrow & \quad T = 147N \end{aligned}$$

The equation of motion for the mass M then gives M_0 as follows

$$M_0 g - T = M_0 a_{1\max} \quad \Rightarrow \quad M_0 = \frac{T}{g - a_{1\max}} = 30kg$$

Now I answer the question asked in the problem.

(a) For 20 kg mass, let the friction between the blocks be f . Then we have

$$f = 5a \quad T - f - f_{2\max} = 10a \quad \text{and} \quad 20g - T = 20a$$

These equations lead to the acceleration of the system as follows

$$20g - f_{2\max} = 35a \quad \Rightarrow \quad a = 3.5ms^{-2}$$

(b) $M = 40kg$. Although I have already shown you that in this case the two blocks will slide on each other. Let me show this to you again in another way. Assume that the blocks move together. In that case the acceleration of the assembly will be

$$a = \frac{40g}{55} = 7.12ms^{-2}$$

But this is larger than the maximum possible acceleration for the 5kg block, so the assembly cannot move together. Under these conditions the equations for the 10kg block and the mass M are

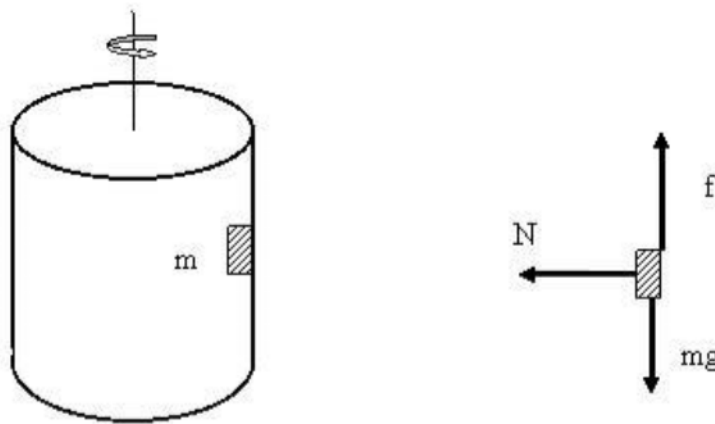
$$T - f_{1\max} - f_{2\max} = 10a \quad \text{and} \quad 40g - T = 40a$$

which gives

$$a = \frac{40g - f_{1\max} - f_{2\max}}{50} = 5.88 \text{ms}^{-2}$$

The 5kg mass of course moves with acceleration of only 4.9ms^{-2} .

Example 2: As the second example let me take a hollow cylinder that is rotating about its axis with a constant angular speed ω . Because of this rotation a mass m on the wall of the cylinder does not slip down (see figure 3). If the coefficient of friction between the cylinder wall and the mass is μ , what is the minimum value of ω for this to happen?



Mass m inside a rotating cylinder. Its free body diagram when it is tuck to the wall of the cylinder (right).

Figure 3

For the mass not to slip, the maximum possible friction on it should be greater than the actual frictional force that holds it against its weight. Since the problem involves rotation we will use cylindrical coordinates. The free body diagram of the mass is as given in figure 3. The mass m experiences three forces when it is stuck to the wall of the cylinder. These forces are its weight mg , the normal reaction N of the cylinder and the frictional force f . In cylindrical coordinates the acceleration of the mass is

$$(\ddot{r} - r\dot{\phi}^2)\hat{r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi} + \ddot{z}\hat{z}$$

so that

$$m(\ddot{r} - r\dot{\phi}^2)\hat{r} + m(r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi} + m\ddot{z}\hat{z} = -N\hat{r} + (f - mg)\hat{z}$$

Now

$$\left. \begin{aligned} r = R = \text{constant} &\Rightarrow \dot{r} = \ddot{r} = 0 \\ z = \text{constant} &\Rightarrow \ddot{z} = 0 \\ \dot{\phi} = \omega \ddot{\phi} = 0 \end{aligned} \right\} \Rightarrow -mR\omega^2 \hat{r} + -N\hat{r} + (f - mg)\hat{z} = 0$$

which gives

$$N = mR\omega^2 \Rightarrow f_{\max} = \mu mR\omega^2 \quad \text{and} \quad f = mg$$

From this minimum angular speed ω_{\min} is calculated as follows.

$$f_{\max} > f \Rightarrow \mu mR\omega^2 > mg \quad \text{or} \quad \omega > \sqrt{\frac{g}{\mu R}}$$

Thus $\omega_{\min} = \sqrt{\frac{g}{\mu R}}.$

So far we have discussed one kind of frictional force where two solid bodies are in contact. We now learn to deal with the drag force which is experienced when a body is moving through gas or a liquid. This force arises due to viscosity of the fluid. To the lowest order in the velocity \vec{v} of the moving body, the drag force is approximated by

$$\vec{F}_{\text{drag}} = -k\vec{v}$$

that is, it is a force in the direction opposite to the velocity and its magnitude is proportional to the speed. So the equation of motion in presence of drag force will read

$$\begin{aligned} m\ddot{\vec{r}} &= \vec{F}_{\text{applied}} - k\vec{v} \\ &= \vec{F}_{\text{applied}} - k\dot{\vec{r}} \end{aligned}$$

If we write it in its component form we have

$$\begin{aligned} m\ddot{x} &= F_x - k\dot{x} \\ m\ddot{y} &= F_y - k\dot{y} \\ m\ddot{z} &= F_z - k\dot{z} \end{aligned}$$

These formulae are valid when the speed of the object is not very large; at large speeds the drag force becomes proportional to the square of the velocity. The simplest example of the effect of

drag is the falling raindrops. Although falling from great heights, they do not hit us with very large speed because of the drag force on them.

As an object falls vertically through a liquid/gas, the drag force on it increases with its speed. At a certain speed - when the drag force equals the weight of the object - it stops accelerating further and therefore moves with a constant speed. This speed is known as the terminal speed or terminal velocity. Assuming drag force to be linearly dependent on velocity, let us estimate the terminal speed of an object when it falls through a liquid of viscosity η . Let the vertically downward direction be y , then

$$m\dot{y} = mg - k\dot{y}$$

But the object will stop accelerating, i.e. $\ddot{y} = 0$, after attaining the terminal speed. Thus at the terminal speed \dot{y}_{term}

$$mg = k\dot{y}_{term}$$

which gives

$$\dot{y}_{term} = \left(\frac{mg}{k} \right)$$

That is the terminal speed of the object. To estimate the terminal speed we need to know what k is. For a spherical object of radius a moving with low speeds, stokes formula gives the drag force to be

$$F_{drag} = -6\pi\eta a v$$

If the object is made of a material of density ρ , the terminal speed comes out to be

$$\dot{y}_{term} = \frac{2}{9} \left(\frac{a^2 \rho g}{\eta} \right)$$

Let us estimate what will be the terminal speed of a rain drop of 2mm radius. With the viscosity of air $\eta = 1.8 \times 10^{-5} \text{ N s / m}^2$, we get

$$\dot{y}_{term} = \frac{2}{9} \times \frac{4 \times 10^{-6} \times 10^3 \times 9.8}{1.8 \times 10^{-5}} \approx 485 \text{ m/s (1750 km/h)}$$

This is too high compared to the observed speeds of about 20 kmph to 5 kmph. Obviously the dependence of drag force on raindrops has higher power dependence on their speeds. In this

lecture we will however restrict ourselves to those cases where the drag force depends linearly on the speed i.e. $\vec{F}_{drag} = -k\vec{v}$. We now solve examples involving such drag force.

Example 3 : An object is thrown in a fluid with initial speed v_0 . Find its speed and the distance traveled by it as a function of time.

Assuming the motion to be in x direction, the equation of motion is

$$m\ddot{x} = -k\dot{x}$$

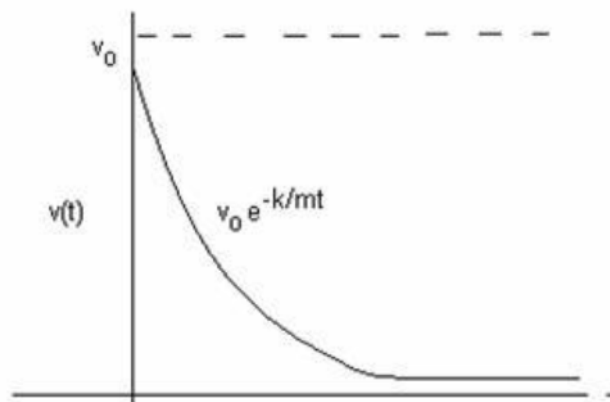
or

$$\frac{d}{dt}(\dot{x}) + \frac{k}{m}\dot{x} = 0$$

You can easily check that the solution is

$$\dot{x} = v_0 e^{-(k/m)t}$$

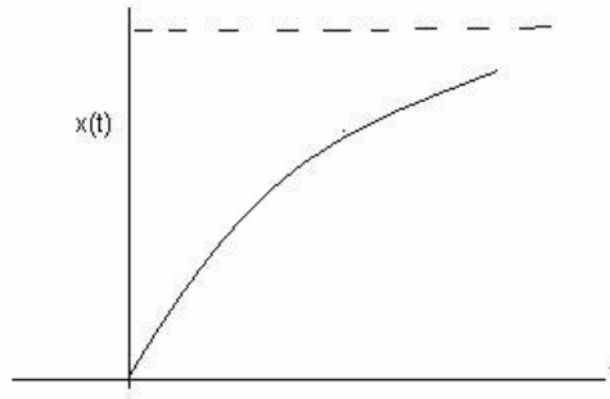
So that the speed initially is v_0 and it decreases exponentially with time. The plot of speed versus time looks like that given below



What about the distance traveled by the object? That is obtained by integrating the speed with respect to time and is

$$\begin{aligned}
 x(t) &= \int_0^t v_0 e^{-(k/m)t'} dt' \\
 &= \frac{mv_0}{k} \left[1 - e^{-(k/m)t} \right]
 \end{aligned}$$

So that the distance traveled looks like



Thus as $t \rightarrow \infty$, the body will stop after traveling a distance of

$$\left(\frac{mv_0}{k} \right)$$

Of course as $k \rightarrow 0$, the distance becomes larger and larger.

Example 4 : We now consider one-dimensional motion of a particle which is moving under the influence of a constant applied force in a medium applying a drag force. Motion of a particle thrown up or falling down is one such example. The equation of motion in this case is

$$\begin{aligned}
 m\ddot{x} &= F - k\dot{x} \\
 \text{or } \frac{d}{dt}(\dot{x}) + \frac{k}{m}\dot{x} &= \frac{F}{m}
 \end{aligned}$$

Let us take the force to be F and the initial speed of the particle to be zero. Without the term

F/m on the right-hand side, the solution of the equation above was $\dot{x} = v_0 e^{-(k/m)t}$ which is, in the language of differential equations, the solution of the homogeneous equation i.e., equation with 0 on the right-hand side. To get the general solution, we add to the homogeneous solution the particular solution corresponding to $F \neq 0$. The particular solution is

$$\dot{x} = \frac{F}{k}$$

So that the general solution for the velocity is

$$\dot{x}(t) = \frac{F}{k} + v_0 e^{-(k/m)t}$$

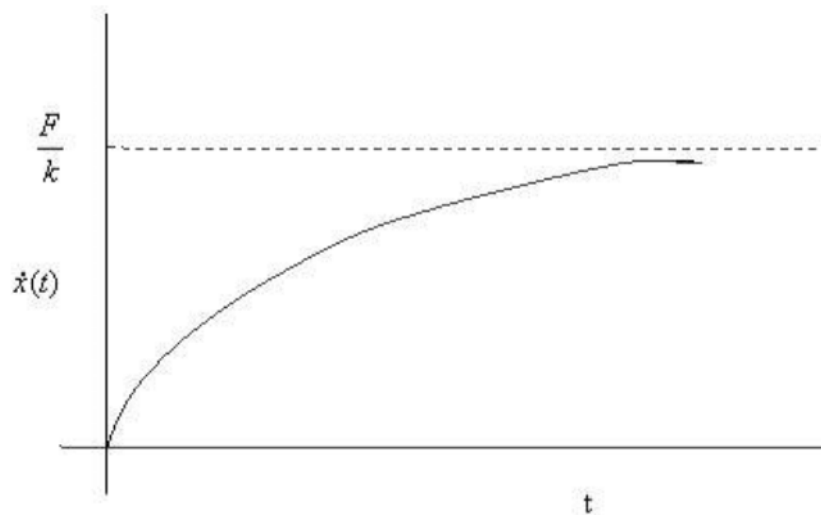
Here v_0 is some constant (not the initial velocity, which is given to be 0). If we start with $v(t=0) = 0$ we get

$$v_0 = -\frac{F}{k}$$

which gives

$$\dot{x}(t) = \frac{F}{k} [1 - e^{-(k/m)t}]$$

The plot of velocity versus time looks as follows



with the terminal speed being $\frac{F}{k}$. The next question we ask is if the solution goes to the standard solution $\left(\frac{F}{m}t\right)$ of particle moving with a constant acceleration when $k=0$. From

$\dot{x}(t) = \frac{F}{k} [1 - e^{-(k/m)t}]$ we get an answer of 0/0 so we have to be careful in taking $k = 0$. Recall that the solution was obtained by assuming $k \neq 0$ because we have been dividing by k . Thus for the $k=0$ case we should take the limit of $k \rightarrow 0$. Doing that we find

$$\begin{aligned}\dot{x}(t) &= \frac{F}{k} \left(1 - 1 + \frac{k}{m}t - \frac{k^2}{m^2}t^2 + \dots\right) \\ &= \frac{F}{m} (t + O(k))\end{aligned}$$

Now $k \rightarrow 0$ gives $\dot{x}(t) = \frac{F}{m}t$ which is the correct answer. We now calculate the distance $x(t)$ traveled by the object as a function of time.

$$\begin{aligned}x(t) &= \int_0^t \dot{x}(t') dt' \\ &= \frac{F}{k} \int_0^t [1 - e^{-(k/m)t'}] dt' \\ &= \frac{F}{k} \left[t - \frac{m}{k} \left(1 - e^{-(k/m)t}\right) \right]\end{aligned}$$

You can see that $t \rightarrow \infty$ the distance is given as

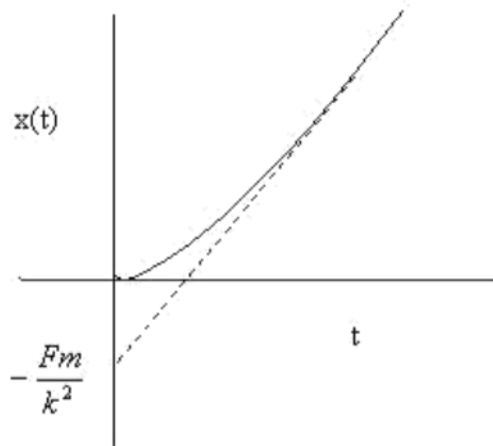
$$x(t) = \frac{F}{k}t - \frac{Fm}{k^2}$$

so at large times it increases linearly with the terminal speed.

For $t \rightarrow 0$ it is

$$\begin{aligned}x(t) &= \frac{F}{k} \left[t - \frac{m}{k} \left(1 - \left(1 - \frac{k}{m}t + \frac{1}{2} \frac{k^2}{m^2}t^2 + \dots\right)\right) \right] \\ &= \frac{F}{k} \left[t - \frac{m}{k} \left(\frac{k}{m}t - \frac{1}{2} \frac{k^2}{m^2}t^2 + \dots\right) \right] \\ &= \frac{1}{2} \frac{F}{m} t^2\end{aligned}$$

This is easily understood as initially there is no drag due to small initial speed and the distance is given by the formula for uniform acceleration. Combining the two limiting cases we see that the plot of $x(t)$ versus time looks like



I'll leave it as an exercise that as $k \rightarrow 0$, we recover the familiar result $x(t) = \frac{1}{2} \frac{F}{m} t^2$. Also I would like you to solve for the velocity and height of a ball thrown up with an initial speed v_0 when drag of air is taken into account.

Next we analyze the effect of drag on the projectile motion in the gravitational field. In this case, we have a projectile shot with initial speed v_0 at an angle θ_0 from the horizontal and we want to find its subsequent motion. The equations of motion are (taking vertically up direction as the y-direction)

$$m\ddot{x} = -k\dot{x}$$

$$m\ddot{y} = -mg - k\dot{y}$$

We have already solved these equations above, so the speed and distance in the x-direction is given as

$$x(t) = v_0 \cos \theta_0 e^{-(k/m)t}$$

$$\text{and } y(t) = \frac{mv_0 \cos \theta_0}{k} [1 - e^{-(k/m)t}]$$

The equation of motion in the y-direction is

$$\frac{d}{dt}(\dot{y}) + \frac{k}{m}\dot{y} = -g$$

Its solution with the initial condition $\dot{y}(0) = v_0 \sin \theta_0$ is

$$\dot{y}(t) = -\frac{mg}{k}[1 - e^{-(k/m)t}] + v_0 \sin \theta_0 e^{-(k/m)t}$$

I give you an exercise now: find at what time $\dot{y} = 0$? Show that this time correctly goes to $\frac{v_0 \sin \theta_0}{g}$

when $k = 0$. Integrating the speed, we get the height $y(t)$ as a function of time. It is given as

$$y(t) = \frac{mv_0 \sin \theta_0}{R}[1 - e^{-(k/m)t}] - \frac{mg}{k} \left\{ t - \frac{m}{k}[1 - e^{-(k/m)t}] \right\}$$

Now to get the trajectory one calculates $x(t)$ and $y(t)$ separately and plots y versus x . I give you some of these for a given v_0, θ_0 but varying k . We take $v_0 = 100\text{m/s}$ and $\theta_0 = 45^\circ$. For no drag situation we get the range $R = 1010\text{m}$ and the highest point of the projectile to be at $h = 254\text{m}$. When a drag coefficient of $k = 0.1$ is introduced we get $R = 495\text{m}$ and $h = 175\text{m}$, a reduction of about 50% in the range and 30% in the height. For $k = 0.2$ we get $R = 313\text{m}$ and $h = 135\text{m}$, giving a further reduction of about 40% in the range and 20% in the height from the corresponding $k = 0.1$ values. Notice when drag force is introduced, the range gets affected much more than the height. The corresponding trajectories are shown below.

One interesting question we may ask is: for zero drag the maximum range is obtained for $\theta = 45^\circ$. If we include drag, should the angle be larger than or less than 45° for obtaining maximum range? Since x-component of the velocity is now decreasing one intuitively feels that the projectile should be given larger speed in the x-direction for maximum range. Thus the projectile should be fired at an angle less than 45° . This is easily understood from the calculations presented above. As we saw in those calculations, for $k \neq 0$ the motion in y direction does not get affected as much as it does in the x -direction. This also suggests that for maximum range we fire the projectile at an angle slightly less than 45° giving it a larger velocity in x -direction. One can also think of it slightly differently. When the particle is shot up drag force is large (because of the initial speed) and also both the gravitational force and drag are working in the same direction. So the particle takes longer to move up the same height than it does in coming down. Since x -velocity is larger in the beginning, the projectile should cover as much distance as possible while ascending than when it is coming down (the x -component may well vanish by that time) This implies that θ_0 should be smaller than 45° .

What we have done so far is to include the simplest form of drag force in solving for the trajectories of motion. However, as the speed increases drag force may also include higher powers of velocity i.e. it may take the form

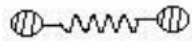
$$\vec{F}_{drag} = -k_1 \vec{v} - k_2 v^2 \hat{v} + \dots$$

where \hat{v} is the unit vector in the direction of the velocity. This is written here to show that force is opposite to the velocity vector. In such cases the corresponding differential equation become non-linear in v and getting the solution becomes difficult, necessitating the use of numerical methods. Some problems though do allow analytic solutions. I end this lecture by giving you one such problem to solve.

Exercise : Throw a ball up with initial velocity v_i and let the force of drag be $= -kv^2$. Find the final speed v_f of the ball when it hits the ground. Also find the height that it goes up to.

UNIT – III
IMPULSE AND MOMENTUM, VIRTUAL WORK
Momentum

So far we have dealt with motion of single particles. Now we are going to make the situation slightly more difficult by letting two or more particles apply forces on one another either by coming in contact or from a distance, and see how we can describe their motion. In such a situation the motion become much more interesting. Let us take an example of only two particles interacting through a spring connected to them, as shown below.



During their motion any of the following could take place: the distance between them may change,



or their orientation may change,



or a combination of both these may occur. Now we wish to develop methods of dealing with such situations. We do this gradually by taking one step at a time. In this regard, we start by introducing the quantity *momentum* that plays a very important role in describing motion when more than one point particle are involved in the motion.

To understand the importance of momentum, let us do the following experiment. Take a cart moving on a frictionless horizontal plane and start putting mass into it; it may be dropped vertically in it (see figure 1 below).

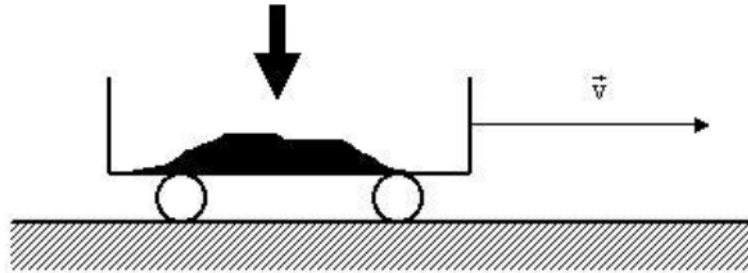


Figure 1

You will see that the cart starts slowing down. If we wish to keep it moving with the same velocity, we find that we have to apply a force on it

$$\vec{F} = \left(\frac{\Delta M}{\Delta t} \right) \vec{v}$$

Compare this with the standard form of Newton's IInd law where we put

$$\vec{F} = M \vec{a} = M \frac{\Delta \vec{v}}{\Delta t}$$

So we see that whether the mass is changed and the velocity kept constant, or the velocity is changed and the mass is kept constant, we have to apply a force to a body. Thus in general

$$\vec{F} = \left(\frac{\Delta M}{\Delta t} \right) \vec{v} + M \frac{\Delta \vec{v}}{\Delta t}$$

(We have ignored the second-order term $\frac{\Delta M \Delta \vec{v}}{\Delta t}$ right now assuming that both the mass and the velocity are varying continuously). Therefore

$$\vec{F} = \frac{\Delta(M\vec{v})}{\Delta t} = \left(\frac{dM\vec{v}}{dt} \right) = \frac{d\vec{p}}{dt}$$

and this defines for us a quality called the momentum denoted above by \vec{p} . By definition

$$\vec{p} = M \vec{v}$$

The force applied on a body or a system of particles is then the rate of change of their total momentum, i.e.

$$\vec{F} = \left(\frac{d\vec{p}}{dt} \right)$$

where \vec{p} now refers to the momentum of the system made up of a collection of particles. In the example taken above, we have to apply a force to keep the cart moving with a constant velocity because as the mass falls in the cart and starts moving with same velocity as the cart, the total momentum of the system - the cart and the mass in it - increases. In writing the definition of the momentum above, we have implicitly assumed that all the particles of the system, with total mass M , are moving with the same velocity. However, if the system is made up of N particles, each one being of different mass $m_i (i = 1 \text{ to } N)$ and also moving with a different velocity \vec{v}_i , the total momentum of the system will be given as

$$\vec{p} = \sum_i m_i \vec{v}_i$$

A fundamental property of momentum is now follows from the definition of force in terms of momentum. If the total force acting on a system of particles is zero, the total momentum of the system does not change with time. To see it clearly let us go back to the two particles connected by a spring (see figure 2 below). There we have

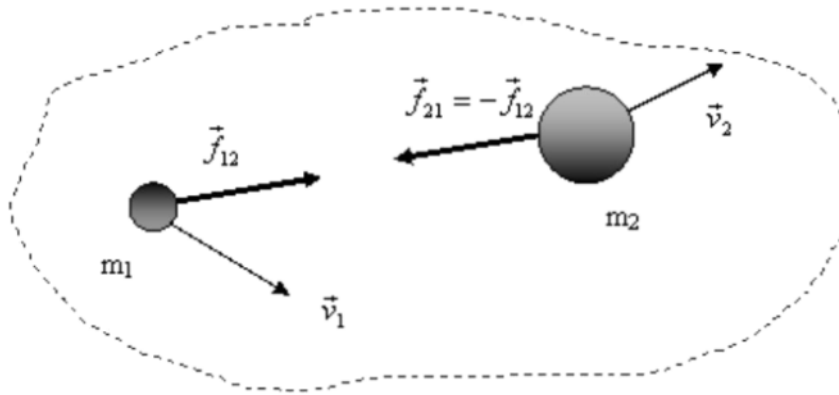
$$m_1 \frac{d\vec{v}_1}{dt} = \vec{f}_{12}$$

for particle 1 and

$$m_2 \frac{d\vec{v}_2}{dt} = \vec{f}_{21}$$

for particle 2. Here \vec{f}_{12} is the force on particle 1 applied by particle 2. Similarly \vec{f}_{21} is the force on particle 2 applied by particle 1. By Newton's third law

$$\vec{f}_{21} = -\vec{f}_{12}$$



Two particles of different masses moving with different velocities. They are applying equal and opposite force on one another.

Figure 2

This immediately results in

$$m_1 \frac{d\vec{v}_1}{dt} + m_2 \frac{d\vec{v}_2}{dt} = \frac{d}{dt} (m_1 \vec{v}_1 + m_2 \vec{v}_2) = 0$$

So no matter how these particles move - their individual velocities \vec{v}_1 or \vec{v}_2 may change - but as long as there is no other force on the system and Newton's third law is obeyed we are going to have

$$m_1 \vec{v}_1 + m_2 \vec{v}_2 = \text{constant}$$

The equation above expresses the principle of momentum conservation - which is a fundamental principle of physics - in its simplest form.

Let us understand this result. If we consider both the particles together as one system, indicated by the dashed line enclosing them in the figure above, there is no force on this system. This is because although each particle is acted upon by a force applied by the other particle, on the system as a whole these two forces act in opposite directions and cancel each other, resulting in a zero net force on the system. As such the momentum of the system does not change. Thus we conclude: ***If the net force acting upon a system of two particles vanishes, their total momentum does not change with time***. Let us now see what happens when we apply forces on each particle also. In that case we have

$$m_1 \frac{d\vec{v}_1}{dt} = \vec{F}_{ext1} + \vec{f}_{12}$$

$$m_2 \frac{d\vec{v}_2}{dt} = \vec{F}_{ext2} + \vec{f}_{21}$$

$$= \vec{F}_{ext1} - \vec{f}_{12}$$

which gives

$$m_1 \frac{d\vec{v}_1}{dt} + m_2 \frac{d\vec{v}_2}{dt} = \vec{F}_{ext1} + \vec{F}_{ext2}$$

or $\frac{d\vec{p}}{dt} = \vec{F}_{total}$

Again we see that no matter how the individual velocities change, the total momentum changes according to the equation

$$\left(\frac{d\vec{p}}{dt} \right) = \vec{F}_{total}$$

Let us now generalize this result to a system of many particles (say N). Then we have for the i^{th} particle

$$m_i \frac{d\vec{v}_i}{dt} = \vec{F}_{exti} + \sum_{j \neq i} \vec{f}_{ij}$$

Where \vec{F}_{exti} is the external force on the i^{th} particle and \vec{f}_{ij} is the force applied on i^{th} particle due to j^{th} particle. Summing it over i gives

$$\sum_i m_i \frac{d\vec{v}_i}{dt} = \sum_i \vec{F}_{exti} + \sum_{\substack{i,j \\ i \neq j}} \vec{f}_{ij}$$

Now we can write

$$\sum_{\substack{i,j \\ i \neq j}} \vec{f}_{ij} = \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} (\vec{f}_{ij} + \vec{f}_{ji})$$

But by Newton's third law $\vec{f}_{ij} = -\vec{f}_{ji}$, which when substituted in the equation above gives

$$\sum_{i \neq j} \vec{f}_{ij} = 0$$

and $\sum_i m_i \frac{d\vec{v}_i}{dt} = \sum_i \vec{F}_{ext i}$

i.e., the total momentum of a system of particles changes due to only the net outside force applied on the system; the interaction between particles does not affect their total momentum.

And if $\vec{F}_{ext} = 0$ i.e., there is no external force on the system,

$$\left(\frac{d\vec{p}}{dt} \right) = 0$$

which means that the total momentum of the system is a constant. That is the statement of conservation of momentum. We will see later that when combined with the principle of conservation of energy, it becomes a powerful tool for solving problem in mechanics. For the time being let us use this principle to develop some intuitive feeling about motion of a collection of particles; looking at it as a single mass.

We now introduce you to the concept of the centre of mass (CM). To do this, let us look at the equation of motion

$$\left(\frac{d\vec{p}}{dt} \right) = \vec{F}_{ext}$$

which is equivalent to

$$\frac{d}{dt} \sum m_i \frac{d\vec{v}_i}{dt} = \vec{F}_{ext i}$$

Since total mass of a collection of particles remains the same, we can divide and multiply the left-hand side of the equation above by the total mass to rewrite it as

$$M \frac{d}{dt} \sum \frac{m_i}{M} \frac{d\vec{v}_i}{dt} = \vec{F}_{ext i}$$

Since $\vec{v}_i = \frac{d\vec{r}_i}{dt}$, where \vec{r}_i is the position of the i^{th} particle, the above equation can also be written as

$$M \frac{d^2}{dt^2} \sum_i m_i \left(\frac{m_i \vec{r}_i}{M} \right) = \vec{F}_{ext (total)}$$

Now we introduce the position vector \vec{R}_{CM} for the centre of mass by writing

$$\vec{R}_{CM} = \frac{m_i \vec{r}_i}{M}$$

so that the equation of motion looks as follows

$$M \frac{d^2 \vec{R}_{CM}}{dt^2} = \vec{F}_{ext(total)}$$

Now we interpret this equation: It says that irrespective of the interaction between the particles and their relative motion, the centre of mass of a collection of particles would always move as if it were a point particle of total mass M moving under the influence of the sum of externally applied forces on each particle, i.e., the total external force. I caution you that the equation above does not imply that all the particles are moving the same way. All it says is that they move in such a way that the motion of their CM is described as if the CM was a particle of mass M .

Let us take an example.

Example 1: Suppose a bomb dropping vertically down explodes in mid air and breaks into three

parts. Let the mass of the bomb be m and those of three pieces $\frac{m}{6}$, $\frac{m}{3}$ and $\frac{m}{2}$, respectively. If the heaviest piece falls $10m$ to the east and the lightest piece $12m$ south of where the unexploded bomb would have dropped, where does the third piece fall?

Since $M\ddot{\vec{R}}_{CM} = \vec{F}_{ext(total)}$, the CM keeps on moving - even after the bomb breaks - vertically down as if it were a point mass of mass M falling under gravity. Thus the CM hits the ground where the unexploded bomb would have fallen. Let us take this point to be the origin with east side being the positive x-axis and the north side the positive y-axis. Then

$\vec{R}_{CM} = 0$, $\vec{r}_{m/2} = 10\hat{i}$, $\vec{r}_{m/6} = -12\hat{j}$ after the bomb pieces having moved for equal times.

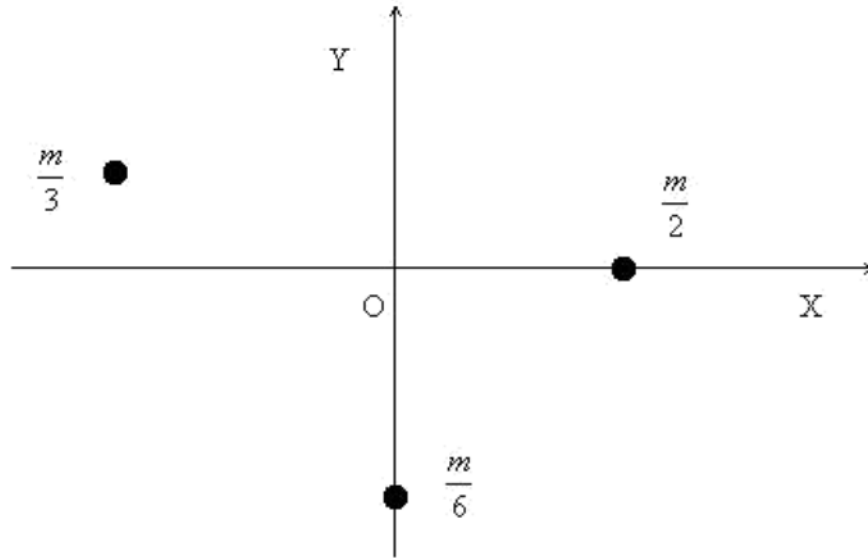
By definition of the centre of mass we have

$$m\vec{R}_{CM} = \frac{m}{6} \times -12\hat{j} + \frac{m}{3} \vec{r}_{m/3} + \frac{m}{2} \times 10\hat{i}$$

With $\vec{R}_{CM} = 0$, this gives

$$\vec{r}_{m/3} = -15\hat{i} + 6\hat{j}$$

Relative positions of the three pieces are shown in figure 3 below, with the centre of mass at the origin.



Positions of the three pieces of the bomb on ground

Figure 3

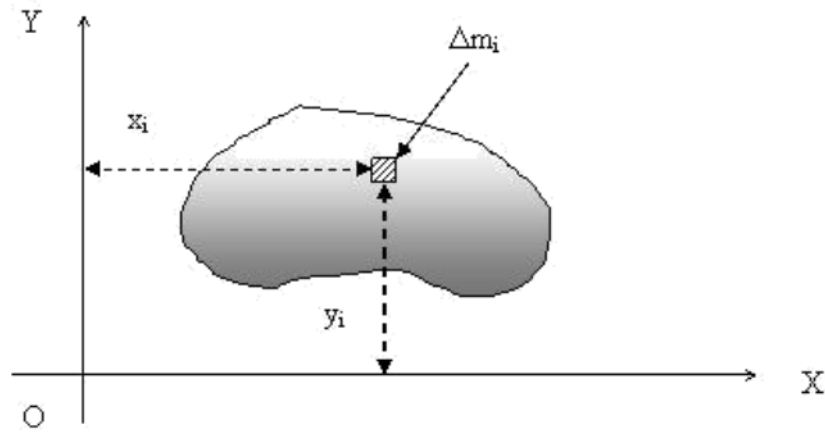
You see that having the knowledge about the position of the other two pieces, we have got the position of the third piece without the knowing anything about the forces generated during the explosion and therefore without solving any equation of motion. That is the power of the momentum conservation principle. I will leave it for you to think which component of momentum is conserved in this case. Would that component be conserved if drag force were included?

Other familiar examples of momentum conservation are a gun recoiling when fired, two persons on roller seats pushing each other and consequently moving away from each other. Look around and you will find many such examples of momentum conservation.

I now discuss a little about calculation of the centre of mass of a mass distribution. Calculation of the centre of mass is similar to calculating the centroid of an area (lecture 7), except that the area is now replaced by mass. For finite masses at given positions, the definition of centre of mass given above is used directly. For a mass distribution in three-dimensions, we calculate all three components of the position of the centre of mass. These are given as

$$X_{CM} = \frac{\sum_i x_i \Delta m_i}{m} = \frac{\int x dm}{m}, \quad Y_{CM} = \frac{\sum_i y_i \Delta m_i}{m} = \frac{\int y dm}{m} \quad \text{and} \quad Z_{CM} = \frac{\sum_i z_i \Delta m_i}{m} = \frac{\int z dm}{m}$$

where dm is a small mass element at the position (x, y, z) in the mass distribution (see figure 4 below).



An element of mass Δm_i and its x - and y -coordinates

Figure 4

We are now going to change the topic a bit and ask how we describe a system where a large force acts for very short durations. A cricket bat striking a ball, a hammer hitting a nail, a person jumping on a floor and coming to sudden stop and a carom striker hitting a coin, or collisions in general, are examples of such forces in operation. In these cases it is not meaningful to talk about the force as a function of time because the time span over which the force acts is very-very short. Further, the force varies a great deal over this short time-interval, as I show in an example below. It is therefore better to describe the overall impact of the force in terms of the momentum change it causes to the system. This is given by the integral of the force over the time that it operates.

Thus $\Delta \vec{p} = \int \vec{F} dt$ describes the effect of the force on the system. The integral $\int \vec{F} dt$ is known as the impulse and denoted by the symbol J . Obviously the momentum change of a system equals the impulse given to it. We now discuss these ideas with the help of an example, that of a ball hitting a wall or any other hard surface.

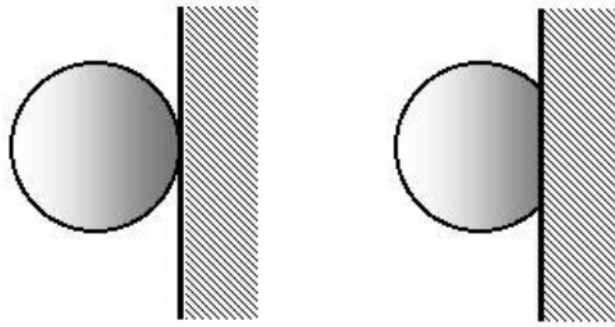
Let us ask what happens when a ball hits a wall or we jump on the floor. If the ball hitting the wall reflects back, that means that the wall has applied a force on the ball so that

$$\Delta \vec{p} = \int \vec{F} dt$$

If the time of contact between the ball and the wall is Δt seconds then the average force is

$$\begin{aligned} \vec{F}_{\text{average}} &= \frac{\Delta \vec{p}}{\Delta t} \\ &= \frac{J}{\Delta t} \end{aligned}$$

But the real force varies greatly from the average force. We show that now. Take the model of the ball as following Hooke's law so that if it is compressed by x by the wall, it applies a force kx on the wall and consequently experiences an equal force in the opposite direction (see figure 5 below).



A ball hitting a wall and getting compressed by amount x

Figure 5

Since the force on the ball follows Hooke's law, the ball performs a simple harmonic motion, its

compression is given by $x = A \sin \omega t$, where A is the maximum compression and $\omega = \sqrt{\frac{k}{m}}$.

From time $t = 0$, when the ball comes in and touches the wall, it takes $t = \frac{\pi}{\omega}$ time (half a cycle) before leaving the wall. The force during this time is given as

$$\begin{aligned} F &= kx \\ &= kA \sin \omega t \end{aligned}$$

$$\omega = \sqrt{\frac{k}{m}} \gg 1$$

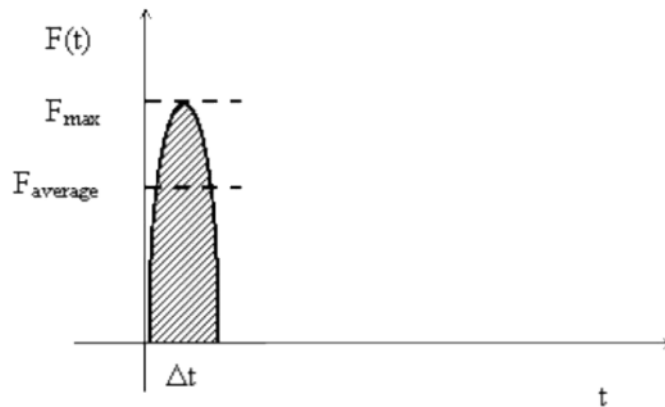
Since for a hard ball k is very large, . So by the time the ball comes back, the force varies with time as shown in the figure 6 below. Here the maximum force F_{max} is given by kA

and $\Delta t = \frac{\pi}{\omega}$. In the figure we show both F_{max} and $F_{average}$. The latter is calculated as

$$F_{average} = \frac{1}{(\pi/\omega)} \int_0^{\pi/\omega} kA \sin \omega t \, dt = \frac{2kA}{\pi}$$

or

$$F_{average} = \frac{2F_{max}}{\pi}$$



Variation of the force on a ball with time as it hits a hard surface at $t=0$, and gets reflected from it Δt time later. The area under the curve gives the impulse. Both F_{max} and $F_{average}$ are also shown.

Figure 6

So you see that over this short period force varies a great deal and is hardly ever near the average force that we calculated. The discussion above has been in terms of a model of the force; the exact force will be different this model and so the variation could be even larger than that shown. It is in such situations, when a strong force is applied over a very short time period, that it is much more meaningful to talk of the total momentum change of a particle than the force

$\vec{F} = \left(\frac{d\vec{p}}{dt} \right)$. Further, in such cases, we generally observe only the initial & final momentum and are hardly concerned about the finer details. It is this change

$$\Delta \vec{p} = \int \vec{F} dt$$

In the momentum that is known as the impulse. So in the ball rebounding from a hard surface with the same speed as it comes in with, the impulse is $-2\vec{p}_i$, where \vec{p}_i is the initial momentum of the ball. So instead of talking of the force applied by the ball on the surface, we say that the ball has imparted momentum to the surface it hit. The amount of momentum transferred is equal to the impulse. This has interesting application in calculating the force on a surface when there are many-many particles continuously hitting a surface, for example molecules in a vessel hitting its walls from inside.

We show two situations in figure 7 below. The upper figure shows the variation of force on a wall when particles hit a surface at some time interval. The lower one, on the other hand, shows

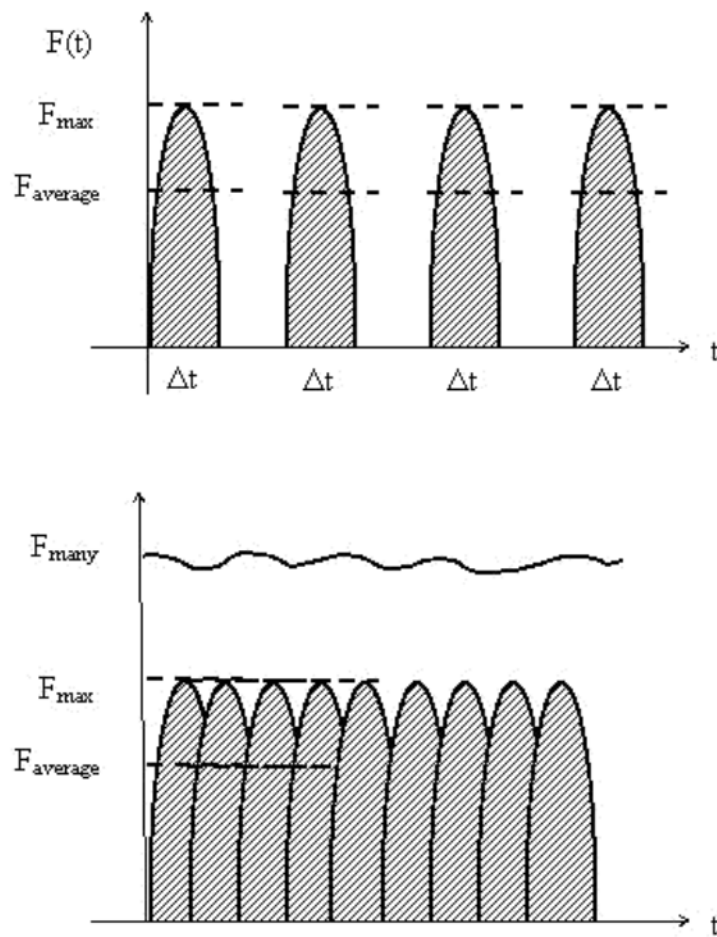
the situation when particles hit continuously. In the first case the force on the surface due to the particles hitting it varies pretty much like the force due to each particle itself. In the second case, however, the force at any instant is given as the sum of the forces applied by each particle at that time. This gives an almost constant force F_{many} as shown in the figure. The value of this force is calculated as follows. Let each particle hitting the surface impart an impulse J to it. If on an average there are n particles per second hitting the surface, then in time Δt the momentum transferred to the surface will be $(n\Delta t)J$. The force F_{many} will then be given as

$$F_{many} = \frac{(n\Delta t)J}{\Delta t}$$

Since $F_{average} = \frac{J}{\Delta t}$, the force above can also be written as

$$F_{many} = (n\Delta t)F_{average}$$

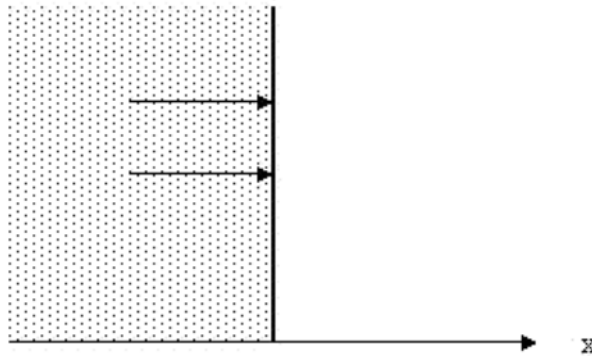
Thus when a stream of particles hits a surface, the force applied by them to the surface equals ***the number of particles striking in time Δt times the average force applied by each one*** of them, a result that you could have anticipated. This is precisely what happens when a jet of water or flowing mass hits another object.



Force on a surface when a stream of particles is hitting it. Upper figure shows the force when particles come one at a time whereas the lower one shows the force when the particles are hitting the surface almost continuously.

Figure 7

As an example let us calculate the pressure of a gas filled in a container. Let the mass of each molecule be m and let their average speed be v . The number density of the molecules in the gas is taken to be n . Now consider a surface of the container perpendicular to the x -axis. (see figure 8).



Molecules of a gas hitting the wall perpendicular to the x-axis.

Figure 8

Each molecule, when reflected from the wall imparts a momentum equal to $2mv_x$ to the wall. The average number of molecules hitting area A of the wall per unit time will be half of those contained in a cylinder of base area A and height v_x (the other half will be moving in the other

direction). This comes out to be $\frac{Av_x n}{2}$. Thus from the formula derived above the force on the wall applied by these molecules is

$$F = Anmv_x^2$$

which gives the pressure

$$p = \frac{F}{A} = nmv_x^2$$

$$= \frac{1}{3}nmv^2$$

This is a result you are already familiar with kinetic theory of gases. But now you know how it comes out. Having done this problem we now deal with another very interesting application of the momentum-force relationship, known as the variable mass problem.

So far we have been dealing with particles of fixed masses. Let us now apply the equation

$\left(\frac{d\vec{p}}{dt}\right) = \vec{F}$ to a problem when the mass of the system under consideration varies with time. The most famous example of this is the rocket propulsion.

Let a rocket with mass M at time t be moving with velocity \vec{v} . A small mass Δm with velocity \vec{u} comes and gets stuck with it so that the rocket now has mass $M + \Delta m$ and moves with a velocity $\vec{v} + \Delta \vec{v}$ (see figure 9 below) after a time interval of Δt . We want to find at what rate does the velocity of the rocket increase? We point out that the word rocket has been used here to represent any system with variable mass.

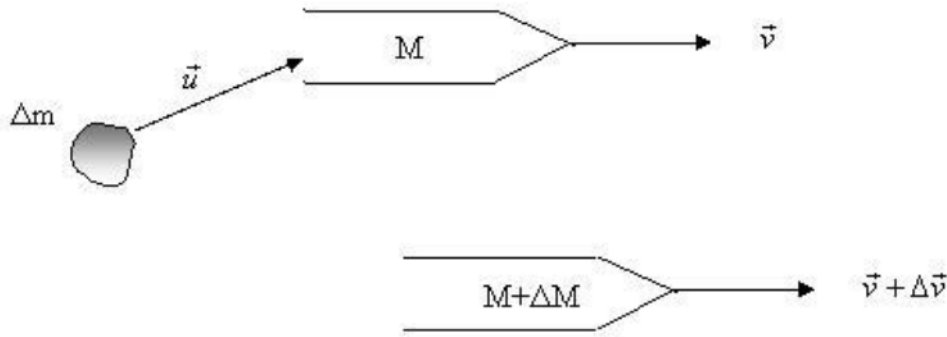


Figure 9

Let us write the momentum change in time interval Δt and equate this to the total external force on the system (that is the sum of external forces acting on M and Δm) times Δt . That gives

$$(M + \Delta m)(\vec{v} + \Delta \vec{v}) - M\vec{v} - \Delta m\vec{u} = \vec{F}_{ext}\Delta t$$

or

$$M\Delta \vec{v} = \Delta m(\vec{u} - \vec{v}) + \vec{F}_{ext}\Delta t - \Delta m\Delta \vec{v}$$

$(\vec{u} - \vec{v})$ is nothing but the relative velocity \vec{u}_{rel} of the mass Δm with respect to the rocket.

Dividing both sides of the equation above by Δt then leads to

$$\frac{M\Delta \vec{v}}{\Delta t} = \frac{\Delta m}{\Delta t}\vec{u}_{rel} + \vec{F}_{ext} - \frac{\Delta m\Delta \vec{v}}{\Delta t}$$

We now let $\Delta t \rightarrow 0$. In this limit $\frac{\Delta m\Delta \vec{v}}{\Delta t}$ also goes to zero for continuously varying mass.

Further, $\frac{dm}{dt} = \frac{dM}{dt}$, the rate of change of the mass of the rocket. Thus the equation for the velocity of a rocket is

$$M \frac{d\vec{v}}{dt} = \frac{dM}{dt}\vec{u}_{rel} + \vec{F}_{ext}$$

Note that both the mass and velocity are now functions of time. For a rocket

$\frac{dM}{dt} < 0$ and $\vec{u}_{rel} < 0$ so that $M \frac{d\vec{v}}{dt} > 0$. It is this term that provides the thrust to the rocket. As pointed out above, *although this equation has been derived keeping rocket in mind, it is true for any system with variable mass*.

Example: We now solve a simple problem involving the rocket equation. A rocket is fired vertically up in a gravitational field. What is its final velocity assuming that the rate of exhaust and its relative velocity remain unchanged during the lift off?

The motion of rocket is one-dimensional. We take the vertically up direction to be positive. Then we have $\vec{u}_{rel} = -u$ where u is a positive number. Therefore the rocket equation takes the form

$$M \frac{dv}{dt} = - \frac{dM}{dt} u - Mg$$

which gives

$$dv = - \frac{dM}{M} u - g dt$$

or

$$v_f = u \ln \left(\frac{M_i}{M_f} \right) - gt_f$$

Here we have taken the initial time and initial velocity both to be zero. Even after the fuel has all been burnt, we see if we observe the rocket time t after being fired, its velocity will be given by the formula

$$v(t) = u \ln \left(\frac{M_i}{M_f} \right) - gt$$

assuming g to be a constant.

Finally, although the momentum-force equation can provide answers for the velocities, I would like to urge you to always think about how the internal forces that generate momenta in opposite directions are generated. That helps in understanding the underlying physics better. For example

in the rocket problems, we say that $\frac{dM}{dt} \vec{u}_{rel}$ provides the thrust to make the rocket move forward. But think about what generates this force? The answer is as follows. In a closed container, gas pressure applies force in all directions and these forces cancel each other. But

when a hole is made from where the gas can escape, the force in the opposite direction is unbalanced; and that is what makes the rocket move. If you understand this, you should be able to answer the following question. If we take a closed box with vacuum inside and punch a hole in it. Which way will it move?

We conclude this lecture by summarizing what we have learnt. We studied the conservation of momentum and a related concept of the centre of mass. Using momentum, we then calculated the force on a surface being hit by a stream of particles, or jet of water. Finally we learnt about the variable mass problem and applied it to a rocket taking off. In the coming lecture we will use the conservation of momentum principle along with the conservation of energy and see how this combination becomes a powerful tool in solving mechanics problems.

UNIT – IV

WORK ENERGY METHOD

Work and Energy

You have been studying in your school that we do work when we apply force on a body and move it. Thus performing work involves both the application of a force as well as displacement of the body. We will now see how this definition comes about naturally when we eliminate time from the equation of motion.

The question that immediately comes to mind is why should we eliminate time from the equation of motion. This is because when we follow the motion of a particle, we are usually interested in velocity as a function of position. Secondly, if we write the equation of motion in terms of time derivatives, it may make the equation difficult to solve. In such cases eliminating time from the equation of motion helps in solving the equation. Let us see this through an example.

Example: Consider the motion of a particle in a gravitational field of mass M . Gravitational force on a mass m is in the radial direction and is given as

$$\vec{F}(\vec{r}) = -\frac{GMm}{r^2} \hat{r}$$

Since the force is in the radial direction, it is better to write the equation of motion in spherical polar coordinates. For simplicity we consider the motion only along the radial direction so that the equation of motion is written as

$$\frac{d^2 r}{dt^2} = -\frac{GM}{r^2}$$

As you can see, integrating this equation to get $r(t)$ as a function of time is very difficult.

On the other hand, let us eliminate time from the equation by using chain rule of differentiation to get

$$\frac{d^2 r}{dt^2} = \frac{d}{dt} \left(\frac{dr}{dt} \right) = \frac{dr}{dt} \frac{d}{dr} \left(\frac{dr}{dt} \right) = v \frac{dv}{dr},$$

where $v = \dot{r}$ is the velocity in the radial direction. This changes the equation of motion to

$$v \frac{dv}{dr} = -\frac{GM}{r^2}$$

This equation is very easy to integrate and gives $v = \dot{r}$ as a function of r , which can hopefully be further integrated to get r as a function of time. Now we go back to what I had said earlier that

the definition of work and energy arises naturally when we eliminate time from the equation of motion. Let us do that first for one dimensional case and analyze the problem in detail.

Work and energy in one dimension

The equation of motion in one-dimension (taking the variable to be x , and the force to be F) is

$$m \frac{d^2 x}{dt^2} = F(x)$$

Let us again eliminate time from the left-hand using the technique used above

$$\frac{d^2 x}{dt^2} = \frac{d}{dt} \left(\frac{dx}{dt} \right) = \frac{dx}{dt} \frac{d}{dx} \left(\frac{dx}{dt} \right) = v \frac{dv}{dx}$$

to get

$$mv \frac{dv}{dx} = \frac{d}{dx} \left(\frac{1}{2} mv^2 \right) = F(x)$$

On integration this equation gives

$$\frac{1}{2} mv_f^2 - \frac{1}{2} mv_i^2 = \int_{x_i}^{x_f} F(x) dx$$

where x_i and x_f refer to the initial and final positions, and v_i and v_f to the initial and final velocities, respectively. We now interpret this result. We define the kinetic energy of a particle of mass m and velocity v to be

$$\text{Kinetic energy} = \frac{1}{2} mv^2$$

and the work done in moving from one position to the other as the integral given above

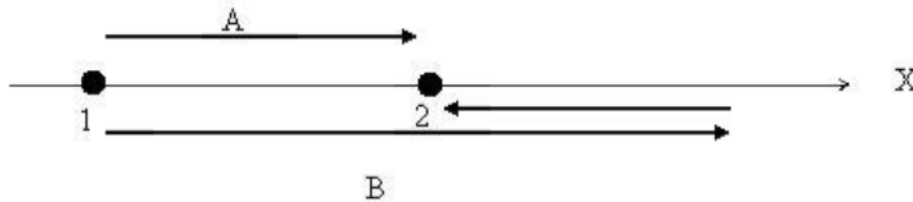
$$\text{Work done} = \int F(x) dx$$

With these definitions the equation derived above tells us that work done on a particle changes its kinetic energy by an equal amount; this known as the *work-energy theorem*.

You may ask: how do we know this equation to be true and consistent with our observations? This is the question that was asked in the early eighteenth century when it was not clear how to define energy, whether as mv or as mv^2 ? The problem with the definition as mv is that if two particles moving in the opposite directions have their energies canceling each other and if they

collide, they stop and all the energy is *lost*. On the other hand, defining it proportional to v^2 makes their energies add up and nothing is lost during collision; the energy just changes form but is conserved. Experimental evidence for the latter was found by dropping weights into soft clay floors. It was found that by increasing the speed of the weights by a factor of two made them sink in a distance roughly four times more; increase in the speed by a factor of three made it nine times more. That was the evidence in favor of kinetic energy being proportional to v^2 .

Potential energy: Let us now define another related energy known as the *potential energy*. This is defined for a force field that may exist in the space, for example the gravitational field or the electric field. Before doing that we first note that even in one dimension, there are many different ways in which one can go from point 1 to point 2. Two such paths are shown in the figure below.



On path A the particle goes directly from point 1 to 2, whereas on path B it goes beyond point 2 and then comes back. The question we now ask is if the work done is always the same in going from point 1 to point 2. This is not always true. For example if there is friction, the work done against friction while moving on path B will be more than on path A. If for a force the work done depends on the path, potential energy cannot be defined for such forces. On the other hand, if the work W_{12} done by a force in going from 1 to 2 is independent of the path, it can be expressed as the difference of a quantity that depends only on the positions x_1 and x_2 of points 1 and 2

(Question: *If the work done is independent of path, what will be the work done by the force field when a particle comes back to its initial position?*). We write this as

$$W_{12} = \int_{x_1}^{x_2} F(x) dx = -U(x_2) + U(x_1)$$

and call the quantity $U(x)$ the potential energy of the particle. We now interpret this quantity. Assume that a particle is in a force field $F(x)$. We now apply a force on the particle to keep it in equilibrium and move it very-very slowly from point 1 to 2. Obviously the force applied by us is $-F(x)$ and the work done by us in taking the particle from 1 to 2, while maintaining its equilibrium, is

$$-\int_{x_1}^{x_2} F(x) dx = -W_{12} = U(x_2) - U(x_1)$$

Thus for a given force field, the potential energy difference $U(x_2) - U(x_1)$ between two points is the *work done by us* in moving a particle, keeping it in equilibrium, from 1 to 2. Note that it is the work done by us - and not by the force field - that gives the difference in the potential energy. By definition, the work done by the force field is negative of the difference in the potential energy. Further, it is the difference in the potential energy that is a physically meaningful quantity. Thus if we want to define the potential energy $U(x)$ as a function of x , we must choose a reference point where we take the potential energy to be zero. For example in defining the gravitational potential energy near the earth's surface, we take the ground level to be the reference point and define the potential energy of a mass m at height h as mgh . We could equally well take a point at height h_0 to be the reference point; in that case the potential energy for the same mass at height h would be $mg(h - h_0)$. Let us now solve another example.

Example: A particle is restricted to move along the x -axis and is acted upon by a force

$$F(x) = \frac{1}{x^2 + a^2}. \text{ Find its potential energy.}$$

We first note that the force is always acting towards the positive x -direction. Thus when we move the particle, we will have to do positive work when taking it towards the negative direction. Thus we expect the potential energy to increase as x becomes more and more negative. By definition

$$\begin{aligned} U(x_2) - U(x_1) &= -\int_1^2 F(x) dx \\ &= -\int_{x_1}^{x_2} \frac{1}{x^2 + a^2} dx \\ &= \tan^{-1}\left(\frac{x_1}{a}\right) - \tan^{-1}\left(\frac{x_2}{a}\right) \end{aligned}$$

Now we choose our reference point. If we choose $U(x_1 = \infty) = 0$, the potential energy is given as

$$U_1(x) = \frac{\pi}{2} - \tan^{-1}\left(\frac{x}{a}\right)$$

On the other hand if we choose $U(x_1 = 0) = 0$, we get

$$U_2(x) = -\tan^{-1}\left(\frac{x}{a}\right)$$

The two energies are shifted with respect to one another by a constant so that the difference in the potential energy between two points is the same for both the forms, as pointed out earlier. The potential energy is lowest for $x = \infty$ and increases as we move towards left and becomes largest for $x = -\infty$. This is precisely what we had anticipated above on the basis of the meaning of potential energy.

Conservation of energy: Having defined potential energy we now combine it with the work energy theorem to come up with another very important conservation principle: that of *conservation of energy*. This is obtained as follows. By the work energy theorem

$$\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = W_{12} = \int_{x_1}^{x_2} F(x) dx$$

and by definition of the potential energy

$$W_{12} = -U(x_2) + U(x_1)$$

Combining the two equations we get

$$\frac{1}{2}mv_2^2 + U(x_2) = \frac{1}{2}mv_1^2 + U(x_1)$$

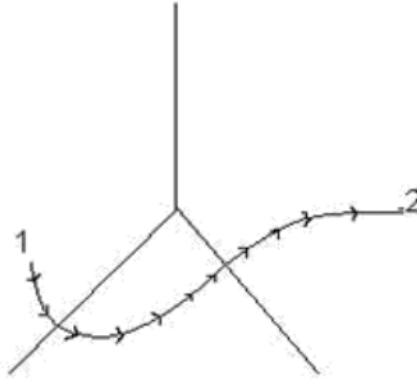
This equation means that if a particle moves in a force field where the work done by the force does not depend on the path taken, the sum of its kinetic and potential energy remains unchanged from one point to another. The sum of the kinetic and potential energy is known as the total mechanical energy. Thus in a force field for which the potential can be defined, total mechanical energy is conserved. Such force fields, where the total mechanical energy is conserved, are therefore known as conservative force fields. Thus whereas the example above is a conservative force field, frictional force is not. **Question:** *If the potential energy is explicitly time-dependent, is the total energy conserved?*

We now move on to generalize and discuss these concepts in three-dimensions.

Work and energy in three dimensions

As we already know, work is defined as the scalar product of the force and displacement vector.

Thus if a particle moves under the influence of a force field $\vec{F}(\vec{r})$ from point 1 to point 2 along the path shown below, the total work is calculated as the sum of partial work done when the particle moves a vanishingly small distance $\Delta\vec{l}$ along the arrows shown below in the figure.



Thus the total work done in gives as

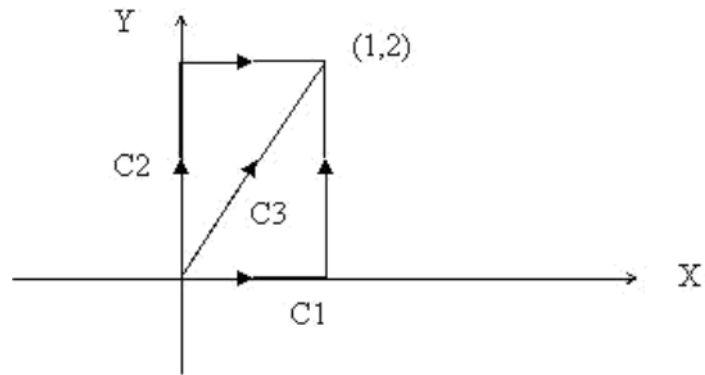
$$W_{12} = \int_{C(12)} \vec{F}(\vec{r}) \cdot d\vec{l}$$

where $C(12)$ indicates that the particle is moving along the curve C from point 1 to 2. Writing the dot product explicitly, we get

$$W_{12} = \int_{C(12)} F_x(x, y, z) dx + \int_{C(12)} F_y(x, y, z) dy + \int_{C(12)} F_z(x, y, z) dz$$

where F_i ($i = x, y, z$) indicates the i^{th} component of the force and x , y and z are varied along the curve. Let us do an example of calculating the work in this manner in two-dimensions.

Example: Consider two force fields (a) $\vec{F}(x, y) = x\hat{i} + y\hat{j}$, and (b) $\vec{F}(x, y) = -y\hat{i} + x\hat{j}$ in the x - y plane. Calculate the work done by these forces when a particle moves from the origin to (1,2) along the three paths C1, C2 and C3 shown in the figure below. On C1 the particle goes along the x -axis first and then moves parallel to the y -axis; on C2 it travels along the y -axis first and then parallel to the x -axis and on C3 it moves along the diagonal.



The work done is given by the formula

$$W_{12} = \int_{C(12)} F_x(x,y,z)dx + \int_{C(12)} F_y(x,y,z)dy$$

Along C1 $y = 0$, $dy = 0$ while moving along the x-axis whereas $x = 1$ and $dx = 0$ when the particle travels parallel to the y-axis. Thus the work done along C1 is

$$W_{12}(C1) = \int_0^1 F_x(x, y=0)dx + \int_0^2 F_y(x=1, y)dy$$

Similarly work done along C2 is given as

$$W_{12}(C2) = \int_0^2 F_y(x=0, y)dy + \int_0^1 F_x(x, y=2)dx$$

For path C3, we have $y = 2x$ so that $dy = 2dx$. Therefore we substitute $y = 2x$ in the functions giving the force and replace dy by $2dx$. As a result, the final integration is over x only with x varying from 0 to 1. Thus the work done is

$$\begin{aligned} W_{12}(C3) &= \int_{C3} F_x(x,y)dx + \int_{C3} F_y(x,y)dy \\ &= \int_0^1 F_x(x, y=2x)dx + 2 \int_0^1 F_y(x, y=2x)dx \end{aligned}$$

We are now ready to work out the work done by force in (a) and (b) (I would like you to plot these force fields and leave it as an exercise for you). For the force in (a) we get

$$W_{12}(C1) = \int_0^1 x dx + \int_0^2 y dy = \frac{5}{2};$$

$$W_{12}(C2) = \int_0^2 y dy + \int_0^1 x dx = \frac{5}{2};$$

$$W_{12}(C3) = \int_0^1 x dx + 2 \int_0^1 (2x) dx = \frac{5}{2}$$

For force (b) on the other hand we get

$$W_{12}(C1) = \int_0^1 (0) dx + \int_0^2 (1) dy = 2$$

$$W_{12}(C2) = \int_0^2 (0) dy + \int_0^1 (-2) dx = -2$$

$$W_{12}(C3) = \int_0^1 (-2x) dx + 2 \int_0^1 (x) dx = 0$$

Thus we see that whereas the force in (a) gives the work to be the same for all three paths, that in (b) gives different work along the three paths. Thus the first force field may be conservative but the second one is definitely not.

Now let us derive the work-energy theorem in three dimensions. Start from the equation

of motion $m \frac{d\vec{v}}{dt} = \vec{F}$ and take the dot product of both sides with the velocity \vec{v} to get

$$m\vec{v} \cdot \frac{d\vec{v}}{dt} = \frac{1}{2} m \frac{d}{dt} (\vec{v} \cdot \vec{v}) = \vec{F} \cdot \vec{v}$$

Now integrate both sides with respect to time and use $d\vec{l} = \vec{v} dt$, where $d\vec{l}$ is the small distance traveled by the particle in time interval dt , to get

$$\begin{aligned} \frac{1}{2} m d(\vec{v} \cdot \vec{v}) &= \frac{1}{2} m d(v^2) = \vec{F} \cdot \vec{v} dt \\ &= \vec{F} \cdot d\vec{l} \end{aligned}$$

On integration this leads to

$$\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = \int_{C(12)} \vec{F} \cdot d\vec{l}$$

This equation tells us that when a force makes a particle move along path C from point 1 to 2, the work done by the force equals the change in its kinetic energy. This is the work-energy theorem in three-dimensions. It is exactly the same as in one dimension except that the work done is calculated by moving along a three-dimensional path.

Potential energy: As is the case in one dimensional motion, potential energy in general can be defined only if the work done is path independent. In that case, the work done depends only on the end points of the path of travel and can be written as the difference on a quantity that is a function of the position vector only. Thus

$$W_{12} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{l} = -U(\vec{r}_2) + U(\vec{r}_1)$$

where $U(\vec{r})$ is defined as the potential energy. Notice that this time I have not written any specific path but just the end points with the integral sign because the work is supposed to be path-independent. From the definition above, it is also evident that here too the difference in the potential energy $U(\vec{r}_2) - U(\vec{r}_1)$ between point 1 and point 2 is the *work done by us* in moving a particle slowly, maintaining its equilibrium, from point 1 to point 2. Now following the exactly same steps that we did for the one dimensional case, we show that

$$\frac{1}{2}mv_2^2 + U(\vec{r}_2) = \frac{1}{2}mv_1^2 + U(\vec{r}_1)$$

Thus when the potential energy can be defined, the total mechanical *energy of a particle is conserved*. I remind you that the total mechanical energy is the sum of the kinetic and the potential energies. In such cases the force is said to be conservative.

By now you may be wondering how can we find out whether a force is conservative or not. Do we have to calculate the work done along all possible paths before we can say that the force is conservative and therefore the principle of conservation of energy holds good. That certainly would be impossible to do. However, there is a much simpler test to check whether a force field is conservative or not. I am going to tell you about it without giving the proof. To find out about the conservative nature of a force $\vec{F}(\vec{r})$, we calculate its curl $\vec{\nabla} \times \vec{F}(\vec{r})$ defined as

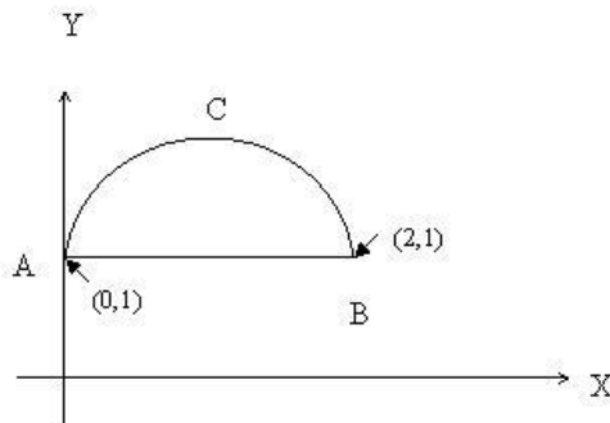
$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Now if the curl of a force field vanishes everywhere, it is a conservative force field. On the other hand if the curl of a force field is nonzero, it is not conservative. Let us now apply this test to the two force fields for which we calculated the work done along different paths. For the force field

$\vec{F}(x, y) = x\hat{i} + y\hat{j}$, the curl is zero everywhere. Hence it is conservative and, as we saw with three paths, the work done in this field is indeed path independent. On the other hand, for

$\vec{F}(x, y) = -y\hat{i} + x\hat{j}$, the curl comes out to be $2\hat{k}$ and therefore the force is not conservative. This was seen above where the work done along the three paths were all different. We now solve an example where knowing the conservative nature beforehand helps us avoid an unnecessary calculation.

Example: Take the force field given by $\vec{F}(x, y, z) = A(x^2 y\hat{i} + \frac{x^3}{3}\hat{j} + z^3\hat{k})$ and consider a particle moving from A to B along the semicircular path ACB (see figure below). Calculate the difference in its kinetic energy at B and at A.



To calculate the change in the kinetic energy of the particle as it moves from A to B, we should calculate the work done by the force in when the particle travels along the semicircle. For this we should calculate

$$W_{ACB} = \int_{(ACB)} F_x(x, y, z=0) dx + \int_{(ACB)} F_y(x, y, z=0) dy$$

withy and d ycalculated from the equation of the circle $(x-1)^2 + (y-1)^2 = 1$. You should try it and see for yourself that the integrals become really lengthy. On the other hand, if the force is conservative, we can calculate the work done in particle moving along the diameter. The latter calculation is much easier. Let us therefore first calculate the curl of the force. It is

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Ax^2y & A\frac{x^3}{3} & Az^3 \end{vmatrix} = 0$$

Thus the work done between any two points is path-independent. We therefore calculate the work along the diameter AB. It is

$$\begin{aligned} W_{AB} &= \int_0^2 F_x(x, y=1, z=0) dx \\ &= \int_0^2 Ax^2 dx \\ &= \frac{4}{3}A \end{aligned}$$

Since the work done is independent of the path, it is going to be the same for the semicircular path ACB also.

After defining the potential energy and getting the principle of conservation energy, we now look a little more at the relationship between the potential energy and the force it gives rise to. As a consequence we also discuss what can we learn about the motion of a particle by looking at its potential energy curve.

Learning about force and motion from the potential energy

We learnt above about how the force leads to the concept of potential energy. However, it is the potential energy that is easier to specify than the force. The reason is very simple: force is a vector quantity and as such in specifying it we have to give its three components as a function of position. On the other hand, potential energy is a scalar quantity and is easier to write as a function of position. For the same reason, many a times it is easier to calculate the potential energy than to calculate the force, as we will see in an example below. Thus generally we give the potential energy of a particle to tell about the force field in which the particle is moving. In this section we discuss what can we learn about the motion of a particle by looking at its potential energy.

First we discuss how do we get the force from the potential energy. Let us first look at one-dimensional case. Employing the definition of potential energy, we find that for a small displacement Δx

$$F(x)\Delta x = -U(x + \Delta x) + U(x) = -\frac{dU(x)}{dx} \Delta x$$

which means that the force is given by the formula

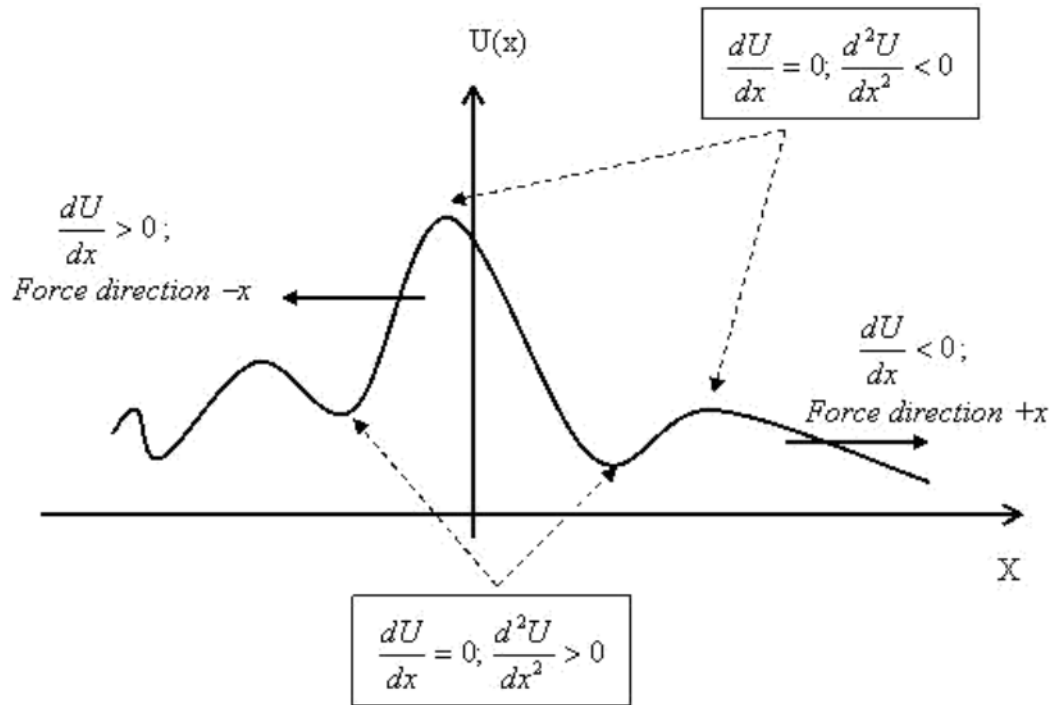
$$F(x) = -\frac{dU(x)}{dx}$$

This is the key formula relating the force to the potential energy. On the basis of this formula, we can infer a lot about the nature of motion by looking at the potential energy curve. First if

$\frac{dU(x)}{dx} > 0$, then the force is towards the negative x-direction and if $\frac{dU(x)}{dx} < 0$, the force is towards the positive x-direction. Thus the force is in the direction of decreasing $U(x)$. What if $\frac{dU(x)}{dx} = 0$? In that case the particle is either on a maximum or a minimum of the potential and there is no force on the particle. The particle is therefore in equilibrium. The equilibrium will be stable one, that is the particle will come back to the equilibrium point when displaced slightly

from that point, if it is at the potential energy minimum or equivalently where $\frac{d^2U(x)}{dx^2} > 0$. On the other hand at the maximum of the potential energy, the particle will rush away from that

point if it is disturbed. Thus at the potential energy maximum, where $\frac{d^2U(x)}{dx^2} < 0$, the equilibrium is unstable. We see that a particle tends to move towards its potential energy minimum and move away from its potential energy maximum. All these concepts can be shown nicely with a bead moving on a smooth frictionless wire bent in the shape of a curve with many maxima and minima and held in the vertical plane (see the figure below). The potential energy of the bead is then proportional to the height of the curve and as such the wire itself represents the potential energy curve in the figure below.



Now with a bead sliding over the wire, you can easily check that all the points made above about the relationship between the force on the bead and the mathematical properties of the potential energy curve are correct. Further the minima and maxima of the curve are clearly observed to be stable and unstable equilibrium points, respectively.

In three dimensions the equivalent of the derivative is the gradient operator. Thus the force $\vec{F}(\vec{r})$ in two or three dimensions is given as

$$\vec{F}(\vec{r}) = -\vec{\nabla}U(\vec{r}) = -\left(\frac{\partial U(x,y,z)}{\partial x}\hat{i} + \frac{\partial U(x,y,z)}{\partial y}\hat{j} + \frac{\partial U(x,y,z)}{\partial z}\hat{k}\right)$$

Thus the force is in the direction opposite to that of increasing U . Further, it vanishes wherever the gradient of the potential energy is zero. Individual components of the force are given as

$$F_x = -\frac{\partial U}{\partial x}; \quad F_y = -\frac{\partial U}{\partial y} \quad \text{and} \quad F_z = -\frac{\partial U}{\partial z}$$

$$\vec{F} = -\frac{\partial U}{\partial x}\hat{i} - \frac{\partial U}{\partial y}\hat{j} - \frac{\partial U}{\partial z}\hat{k}$$

A word of caution is needed here. does not mean that if we transform to some other co-ordinates system (say spherical) then

$$\vec{F} = -\frac{\partial U}{\partial r} \hat{r} - \frac{\partial U}{\partial \theta} \hat{\theta} - \frac{\partial U}{\partial \phi} \hat{\phi}$$

will be correct. This is not even dimensionally correct. To get the correct answer, one must properly transform from Cartesian to polar co-ordinates. The result then is

$$\vec{F} = -\frac{\partial U}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial U}{\partial \theta} \hat{\theta} - \frac{1}{r \sin \theta} \frac{\partial U}{\partial \phi} \hat{\phi}$$

Thus in spherical polar coordinate system, the force components are given as

$$F_r = -\frac{\partial U}{\partial r}, \quad F_\theta = -\frac{1}{r} \frac{\partial U}{\partial \theta} \quad \text{and} \quad F_\phi = -\frac{1}{r \sin \theta} \frac{\partial U}{\partial \phi}$$

Similarly in cylindrical coordinate system the force is related to the potential energy as

$$\vec{F} = -\frac{\partial U}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial U}{\partial \phi} \hat{\phi} - \frac{\partial U}{\partial z} \hat{z}$$

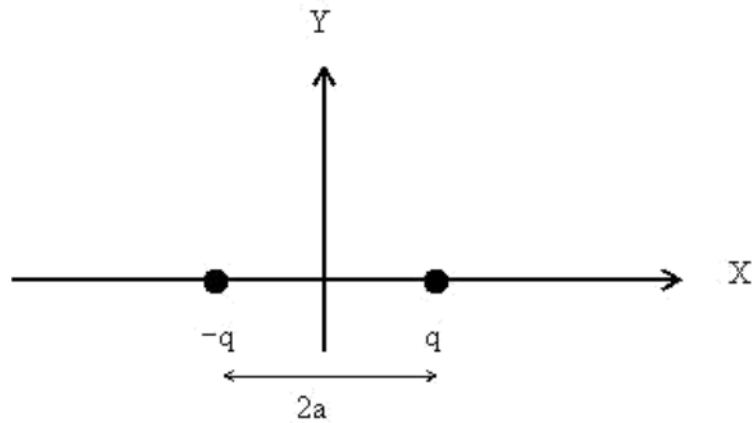
With the individual force components

$$F_r = -\frac{\partial U}{\partial r}, \quad F_\phi = -\frac{1}{r} \frac{\partial U}{\partial \phi} \quad \text{and} \quad F_z = -\frac{\partial U}{\partial z}$$

Having given you the prescription for obtaining force from the potential energy let us now apply it to find the field of an electric dipole using its scalar potential.

Example: As an application of finding force from the potential, let us calculate the electric field due to a dipole.

Let the dipole be situated at the origin along the x-axis. Let the charges $-q$ and $+q$ be separated by distance $2a$ (see figure below) so that the dipole moment is $\vec{p} = 2qa\hat{i}$. Then potential and field at any point can be calculated by adding the field due to the two charges. Adding the field in this case becomes a bit difficult because we have to obtain three components of the field for each charge and add them. On the other hand, finding the potential is relatively easy because it is a scalar quantity and we obtain it by adding the potential due to two charges. Then the gradient gives the field. In the calculation we assume that $a \rightarrow 0$ and q is correspondingly very large so that their product is finite. We will be using this by keeping term only linear in a and neglecting higher orders.



The potential (potential energy per unit charge) due the two charges is given as

$$\begin{aligned}
 U(x, y, z) &= \frac{kq}{\sqrt{(x-a)^2 + y^2 + z^2}} - \frac{kq}{\sqrt{(x+a)^2 + y^2 + z^2}} \\
 &= \frac{kq}{\sqrt{x^2 + y^2 + z^2 - 2ax}} - \frac{kq}{\sqrt{x^2 + y^2 + z^2 + 2ax}} \\
 &= \frac{kq}{r} \left(1 + \frac{xa}{r^2} \right) - \frac{kq}{r} \left(1 - \frac{xa}{r^2} \right) = \frac{2kqax}{r^3} \\
 &\text{(Keeping terms up to order } a \text{)} \\
 &= \left(\frac{kp}{r^3} x \right)
 \end{aligned}$$

Now taking the gradient we get the three components of the force, which are

$$\begin{aligned}
 F_x &= -\frac{\partial U}{\partial x} = -\frac{\partial}{\partial x} \left(\frac{kp x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right) \\
 &= -\frac{kp}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} + \frac{3}{2} \frac{2kp x^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \\
 &= \frac{3kp x^2}{r^5} - \frac{kp}{r^3}
 \end{aligned}$$

Similarly

$$F_y = -\frac{\partial}{\partial y} \left(\frac{kpx}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right) = \frac{3}{2} \left(\frac{2kpxy}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \right) = \frac{3kpxy}{r^5}$$

$$F_z = -\frac{\partial}{\partial z} \left(\frac{kpx}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right) = \frac{3}{2} \left(\frac{2kpxz}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \right) = \frac{3kpxz}{r^5}$$

Combining these results together we get for the field of the dipole

$$\vec{E}(\vec{r}) = k \frac{1}{r^3} [3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}]$$

I would like you to get the same result by adding the fields of the charges together and compare the answers.

In these lectures, we have learnt: the work-energy theorem, definition of potential and its relationship with the force field, concept of conservative forces and the principle of conservation of energy. I leave these lectures by giving you a few exercises.

Exercise 1: Consider one-dimensional motion in a potential $U(x)$. Show that if a particle of mass m is displaced slightly from its equilibrium position at a potential energy minimum at x_0 , it will perform simple harmonic oscillations. Find the corresponding frequency.

Exercise 2: Consider two different inertial frames moving with respect to one another with a constant velocity. Starting from the work-energy theorem in one frame, prove that it is true in the other frame also.

Collisions

In the previous two lectures, we have seen that when many particles are interacting, there are two conservation systems that are obeyed by them. One, if the net external force on the particles is zero, the total angular momentum of the system remains a constant. This is expressed mathematically as

$$\sum_i \vec{f}_{ext i} = 0 \Rightarrow \sum_i m_i \vec{v}_i = \text{constant}$$

$$\text{or } \frac{d}{dt} \left(\sum_i m_i \vec{v}_i \right) = 0$$

Further we saw during the motion of a many particle system, one point - its centre of mass - moves as if its mass M is equal to the total mass $\sum_i m_i$ of the system and the total force $\vec{F}_{ext} = \sum_i \vec{f}_{exti}$ is being applied on that mass. The CM co-ordinate is defines as

$$\vec{R}_{CM} = \frac{\sum_i m_i \vec{r}_i}{M}$$

And it moves according to the equation

$$M\ddot{\vec{R}}_{CM} = \vec{F}_{ext}$$

Thus if $\vec{F}_{ext} = 0$ then $\dot{\vec{R}}_{CM} = \text{constant}$. That means if the total external force on the system is zero, the CM moves with a constant velocity. This is another way of expressing the conservation of linear momentum.

The other conservation principle that we saw was that of total energy. Accordingly the total energy, which is the sum of their kinetic energy KE_i and potential energy PE_i , of a system of particles remains a constant

$$\sum_i (KE_i + PE_i) = \text{constant}$$

As an example of the power of these principles, in this lecture we apply these two principles to the problem of two particles of masses m_1 and m_2 colliding.

Before we discuss the problem of two particles colliding, we prove something very important and useful: *Kinetic energy of a system of particles is equal to the sum of the kinetic energy of its centre of mass and kinetic energy of particles with respect to the centre of mass*. By kinetic energy of the CM we mean its kinetic energy calculated as a point particle of the total mass

$M = \sum_i m_i$ moving with the velocity $\dot{\vec{R}}_{CM}$ of the CM. To see this, substitute in the expression for the kinetic energy

$$\frac{1}{2} \sum_i m_i \vec{v}_i^2$$

$\vec{v}_i = \vec{v}_{CM} + \vec{v}_{ic}$, where \vec{v}_{CM} is the velocity of the CM and \vec{v}_{ic} is the velocity of i^{th} particle in the CM frame. This gives

$$\begin{aligned}
 KE &= \frac{1}{2} \sum_i m_i (\vec{v}_{CM} + \vec{v}_{ic})^2 \\
 &= \frac{1}{2} \left(\sum_i m_i \right) \vec{v}_{CM}^2 + \frac{1}{2} \sum_i m_i \vec{v}_{ic}^2 + \vec{v}_{CM} \cdot \left(\sum_i m_i \vec{v}_{ic} \right)
 \end{aligned}$$

Now $\sum_i m_i \vec{v}_{ic}$ is the momentum of the CM with respect to the CM and therefore proportional to the velocity of the CM with respect to the CM. But the velocity of the CM relative to the CM is

zero implying that $\sum_i m_i \vec{v}_{ic} = 0$. This immediately gives

$$\begin{aligned}
 KE &= \frac{1}{2} M \vec{v}_{CM}^2 + \frac{1}{2} \sum_i m_i \vec{v}_{ic}^2 \\
 &= KE \text{ of the CM} + KE \text{ about the CM}
 \end{aligned}$$

This result, that the kinetic energy of a system of particles can be decomposed into KE of the CM and KE about the CM, is very important and useful. In a later lecture, we will see that the same is true for the angular momentum.

The division of kinetic energy as shown above is useful in learning how energies are shared when particles interact with each-other for short periods of time. As an example take explosion of a bomb. Since the CM will keep on moving the same way as it was before the explosion - because the forces generated are between the pieces of the bomb and therefore have no effect on the total momentum of the system - the explosion does not change the kinetic energy of the CM. Thus all the energy released in the explosion goes to the kinetic energy of the pieces of the bomb with respect to the CM. As another example, consider two particles colliding and getting stuck together. Since the CM keeps on moving with the same speed because of momentum conservation, the minimum kinetic energy that the masses stuck together have to have is that of the centre of mass. Thus the maximum possible energy loss in this case is the sum of their kinetic energy relative to their CM (also called the kinetic energy in the CM frame).

We now get back to the problem of two particles colliding. We consider two particles of masses m_1 and m_2 coming in with velocities \vec{v}_1 and \vec{v}_2 , respectively, interacting in a region, and then going out with velocities \vec{v}'_1 and \vec{v}'_2 (see figure 1). This is the simplest collision problem. If more particles are involved then the problem is going to be more complicated.

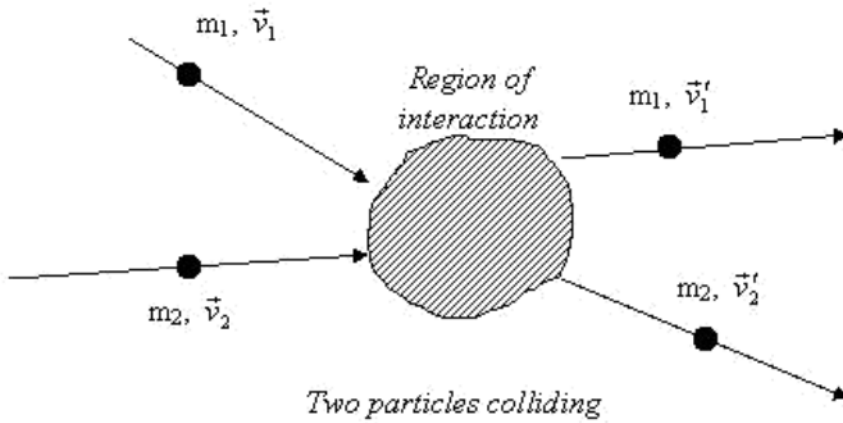


Figure 1

Since we assume particles interact only when they are close to each other, they are essentially free before and after the collision. Further, the interaction region is very small; thus even if the particles are in an external field, the potential energy remains essentially unchanged during the collision. Thus we can write

$$\frac{1}{2} m_1 \vec{v}_1^2 + \frac{1}{2} m_2 \vec{v}_2^2 + \Delta E = \frac{1}{2} m_1 \vec{v}_1'^2 + \frac{1}{2} m_2 \vec{v}_2'^2$$

where we have added ΔE on the left-hand side to take into account any addition or loss of energy during the interaction of particles. For example if the particles generate some energy during interaction, $\Delta E > 0$. This will be the case when two particles release some chemical energy. On the other hand, $\Delta E < 0$ when the particles lose energy during interaction. This is called an inelastic collision. $\Delta E = 0$ is the case of elastic collision; here the total kinetic energy before and after the collision is the same. If particles interact over a large region, we can take the velocities to be in the asymptotic region, where the particles are far apart and therefore the equations above are applicable. The discussion so far has been in terms of balancing the energies involved during the interaction.

The other conservation principle is that of conservation of momentum. Usually during collision the impulse due to collision (internal force if two particles are considered as one system) is much larger than any external impulses. So we neglect it and conserve momentum. If the external impulse comparable to the internal impulse, it must be taken into account. This could be the case when the external force is very large or the particles interact for a long time. For the time being though, we will focus on cases where external impulse can be neglected. Thus

$$m_1 \vec{v}_1 + m_2 \vec{v}_2 = m_1 \vec{v}_1' + m_2 \vec{v}_2'$$

The two equations are actually a set of four equations with momentum conservation giving three equations, one for each component. However, given \vec{v}_1, \vec{v}_2 and ΔE , we have to solve for six quantities, three components for \vec{v}_1' and three for \vec{v}_2' . Thus to solve the problem completely, we

need more information, for example the scattering angles. In two dimensions also, the conservation equations alone are not enough to solve the problem of finding velocities after the collision. This is because now there will be four unknowns - two components for velocity of each particle - but only three equations, one from the energy balance and two from momentum conservation. Only in one dimension, we can solve the collision problem completely because there are two equations and two unknowns. Nonetheless, we can get a lot of information about the motion from these two conservation laws as we now discuss.

As the first example, let us consider two particles of masses m_1 and m_2 moving with velocities \vec{v}_1 and \vec{v}_2 , respectively, colliding, getting stuck together to make a particle of mass $(m_1 + m_2)$ that moves with velocity \vec{V} . In the process energy ΔE is released. Then momentum conservation tells us

$$m_1 \vec{v}_1 + m_2 \vec{v}_2 = (m_1 + m_2) \vec{V}$$

and balancing the energy gives

$$\frac{1}{2} m_1 \vec{v}_1^2 + \frac{1}{2} m_2 \vec{v}_2^2 + \Delta E = \frac{1}{2} (m_1 + m_2) \vec{V}^2$$

Notice that we have added to ΔE to the left-hand side so that the total final kinetic energy is the sum of the total initial kinetic energy and the energy added to the system. Substituting for \vec{V} from the momentum conservation equation in the energy equation, we get

$$m_1 \vec{v}_1^2 + m_2 \vec{v}_2^2 + 2\Delta E = \frac{(m_1 + m_2)(m_1 \vec{v}_1 + m_2 \vec{v}_2)^2}{(m_1 + m_2)^2}$$

which on simplification gives

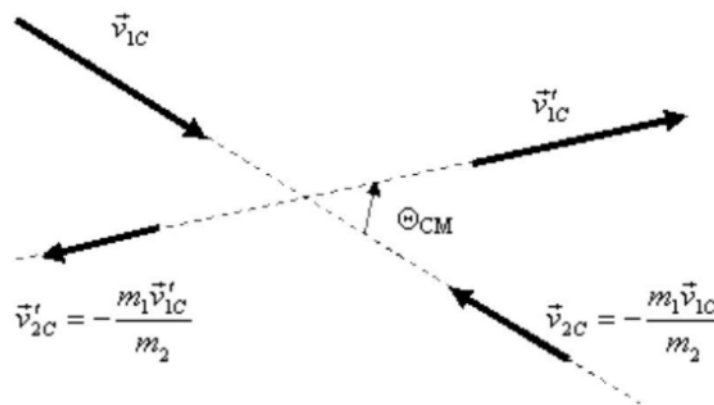
$$(\vec{v}_1 - \vec{v}_2)^2 = -\frac{2\Delta E(m_1 + m_2)}{(m_1 + m_2)}$$

The left-hand side of the equation above is definitely positive. On the other hand, the right-hand side is negative if $\Delta E > 0$, i.e., the final kinetic energy is larger than the initial kinetic energy. So this reaction will not be possible if it is exothermic, i.e., some energy is generated and added to the initial kinetic energy. Thus two atoms colliding in free space will not combine to form a molecule (in which process the energy is usually released). However if energy is taken away from the system, i.e. $\Delta E < 0$, then the reaction is possible. This is the information we have got purely on the basis of conservation laws. We now go on to discuss collisions as described with respect to the CM. We will see that this gives us a lot of insight into the collision problem.

As we had stated earlier, the conservation of momentum implies that the centre of mass moves with a constant velocity when there is no external force on the particles. Thus if we attach a

frame to the CM, it will also move with constant velocity and will be an initial frame of reference. Let us call this the CM frame. Since it is an inertial frame, we can equally well describe a collision process in a CM frame. Observing a collision from the CM frame gives us the biggest advantage that the sum of the momenta (the total momentum) is always zero in this frame. In this lecture we will be focusing on two particle collisions as described from the CM frame. We will see that because of the total momentum being zero, description of a collision in this frame becomes simpler. In coming lectures we will see that CM provides a convenient origin for studying rotational motion also.

For now, let us look at the two particles collision. As stated above, in the CM frame the total momentum is always zero because in this frame the CM does not move. So that the velocities of two particles in the CM frame are always in the direction opposite to each other. Further the motion remains confined to a plane formed by the lines representing the initial and the final velocities directions (keep in mind that the velocities of the two particles at any instant are along the same line though opposite in direction). Thus in the CM frame a collision looks as shown in figure 2.



A two particle collision observed from the CM frame. Incoming velocities are shown by unprimed symbols whereas outgoing velocities are primed. Scattering angle is Θ_{CM} .

Figure 2

In figure 2 two particles with masses m_1 and m_2 and velocities \vec{v}_{1C} and $\vec{v}_{2C} = -\frac{m_1 \vec{v}_{1C}}{m_2}$ are coming in for a collision; they collide and particle 1 goes out with velocity \vec{v}'_{1C} and particle 2 with $\vec{v}'_{2C} = -\frac{m_1 \vec{v}'_{1C}}{m_2}$. In the process particle 1 gets deflected by an angle Θ_{CM} . As stated earlier, even in 2d there are four unknowns: two components of \vec{v}'_{1C} and two of \vec{v}'_{2C} to be obtained but only three equations- one for energy conservation and two for momentum conservation. So the problem cannot be solved fully by using conservation principle only. However, if the interaction is

known, then Θ_{CM} and both the velocities after collision can in principle be calculated. Let us now see how much can we learn about the motion after collision applying only the conservation principles. We will be discussing both the elastic and inelastic collisions. Recall that if the kinetic energy remains unchanged in a collision, the collision is elastic; on the other hand, if the energy is lost the collision is inelastic.

Let us first focus on an elastic collision and analyze it in the CM frame. As pointed out earlier, the velocities of the two particles before and after collision are opposite to each other. Thus the relationship between the magnitudes v_{1C} , v_{2C} , v'_{1C} and v'_{2C} of the velocities is

$$m_1 v_{1C} = m_2 v_{2C}$$

$$m_1 v'_{1C} = m_2 v'_{2C}$$

$$\frac{1}{2} m_1 v_{1C}^2 + \frac{1}{2} m_2 v_{2C}^2 = \frac{1}{2} m_1 v_{1C}'^2 + \frac{1}{2} m_2 v_{2C}'^2$$

Substituting for v_{2C} and v'_{2C} from the first two equations in the last one we get

$$v_{1C}' = v_{1C} \quad \text{and} \quad v_{2C}' = v_{2C}$$

Thus the velocity vectors of both particles just rotate but do not change in magnitude as the particles move out after collision. You have learnt in previous classes that in an elastic collision the magnitude of the relative velocity of one particle with respect to the other remains unchanged during the collision. In one dimension it means that the speed of approach of two particles is the same as their speed of separation. Let us now see how it follows directly from the conservation principles.

As we have derived above, $v_{1C}' = v_{1C}$ and $v_{2C}' = v_{2C}$ in an elastic collision. If the velocities of the two particles are \vec{v}_1 and \vec{v}_2 , respectively, in the ground frame, then

$$\vec{v}_1 = \vec{v}_{CM} + \vec{v}_{1C} \quad \text{and} \quad \vec{v}_2 = \vec{v}_{CM} + \vec{v}_{2C}$$

Similar relationships hold for the velocities after collision i.e.

$$\vec{v}_1' = \vec{v}_{CM} + \vec{v}_{1C}' \quad \text{and} \quad \vec{v}_2' = \vec{v}_{CM} + \vec{v}_{2C}'$$

Using these relationships we find that

$$\begin{aligned}
 (\vec{v}_2 - \vec{v}_1)^2 &= (\vec{v}_{2C} - \vec{v}_{1C})^2 \\
 &= v_{2C}^2 + v_{1C}^2 - 2\vec{v}_{2C} \cdot \vec{v}_{1C} \\
 &= v_{2C}^2 + v_{1C}^2 + 2v_{2C} \cdot v_{1C} \\
 &\quad (\text{because } \vec{v}_{2C} = -\vec{v}_{1C})
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 (\vec{v}'_2 - \vec{v}'_1)^2 &= (\vec{v}'_{2C} - \vec{v}'_{1C})^2 \\
 &= v_{2C}'^2 + v_{1C}'^2 - 2\vec{v}'_{2C} \cdot \vec{v}'_{1C} \\
 &= v_{2C}'^2 + v_{1C}'^2 + 2v_{2C}' \cdot v_{1C}' \\
 &\quad (\text{because } v_{1C}' = v_{1C}, v_{2C}' = v_{2C} \text{ and } \vec{v}'_{2C} = -\vec{v}'_{1C})
 \end{aligned}$$

Thus we see that in an elastic collision

$$(\vec{v}'_2 - \vec{v}'_1)^2 = (\vec{v}_2 - \vec{v}_1)^2$$

We have shown that the magnitude of relative velocity of one particle with respect to other remains the same in an elastic collision.

To see the dramatic effects of a nearly elastic collision, take a table-tennis ball (very small mass m), put it on a large bouncy ball of mass M ($M \gg m$), and drop them from a height (see figure 4) on a hard floor. You will see that the table-tennis ball bounces back really high after the balls hit the ground. Can you work how high will it go if the balls are dropped from a height h ? Assume that no energy is lost.

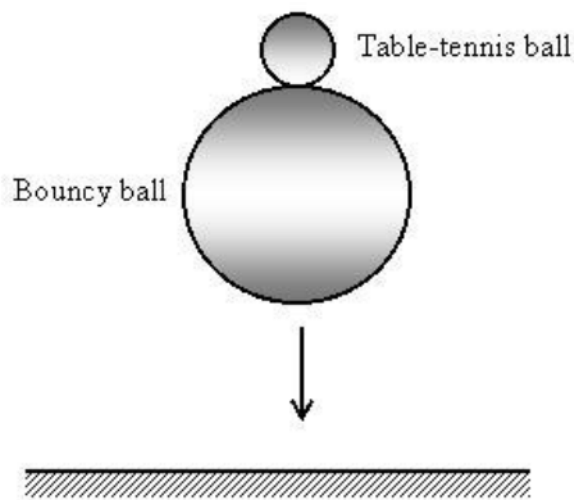
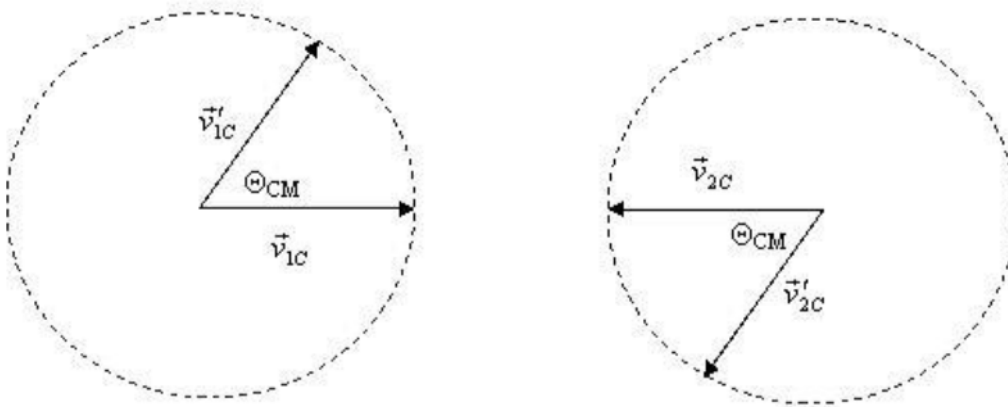


Figure 4

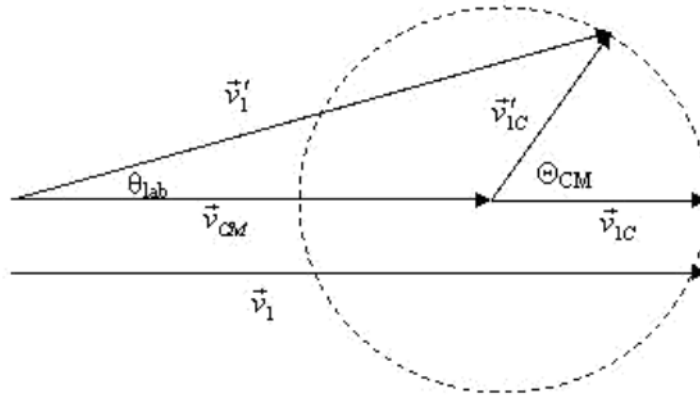
Now we consider a two-particle elastic collision in a plane and analyze it. This could be the collision of a striker and a coin on a carom board, for example. It is a two-dimensional case. We are going to analyze the motion graphically. First we look at the velocities in the CM frame. If we take the initial direction of particle 1 towards $+x$, the velocities of the two particles before and after collision can be shown as done in figure 5. Keep in mind that in an elastic collision, the magnitude of the velocities of each particle remains unchanged in the CM frame. However the direction of the velocity for each particle changes by an angle θ_{CM} as shown in figure 5.



Velocities of two colliding particles before and after collision as seen in the CM frame

Figure 5

The picture above shows the angle of scattering in the CM frame. However experiments are done on ground - and not in the CM frame. So we should be answering the question: by what angle θ_{lab} does particle 1 scatter in the laboratory frame? Since velocities \vec{v}_1 and \vec{v}'_1 in the lab frame are given as $\vec{v}_1 = \vec{v}_{CM} + \vec{v}_{1C}$ and $\vec{v}_2 = \vec{v}_{CM} + \vec{v}_{2C}$, the relationship between these velocities can be shown as done in figure 6.



Velocities of particle 1 before and after collision as seen in the CM and the lab frame.

Figure 6

From figure 6, it is now very easy to see that

$$\begin{aligned}\tan \theta_{lab} &= \frac{v_{1C}' \sin \theta_{CM}}{v_{CM} + v_{1C}' \cos \theta_{CM}} \\ &= \frac{v_{1C} \sin \theta_{CM}}{v_{CM} + v_{1C} \cos \theta_{CM}} \\ &= \frac{\sin \theta_{CM}}{(v_{CM}/v_{1C}) + \cos \theta_{CM}}\end{aligned}$$

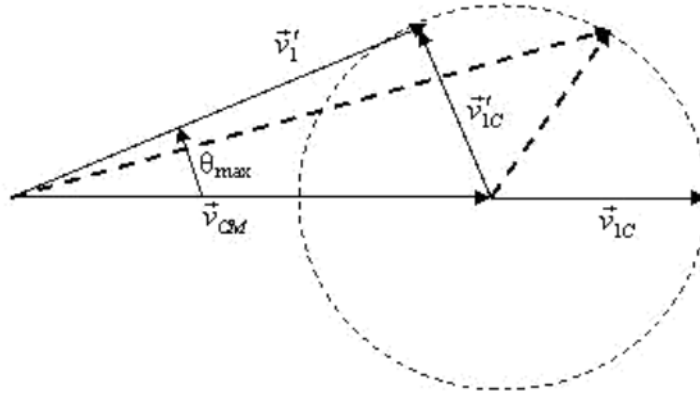
Similar relationships can also be derived for particle 2. Now if particle 2 was at rest when hit by particle 1, then

$$v_{CM} = \frac{m_1 v_1}{m_1 + m_2} \quad \text{and} \quad v_{1C} = \frac{m_2 v_1}{m_1 + m_2}$$

This gives

$$\tan \theta_{lab} = \frac{\sin \theta_{CM}}{(m_1/m_2) + \cos \theta_{CM}}$$

Let us now look at two cases: $m_1 > m_2$ and $m_1 < m_2$. In the case of $m_1 > m_2$, θ_{lab} cannot be greater than a particular angle θ_{max} . This can be either calculated by using the expression above or alternatively, graphically as we do. For $m_1 > m_2$ we also have $v_{CM} > v_{1C}$. Thus a picture showing the velocities in the laboratory and the CM frame looks like that in figure 7.



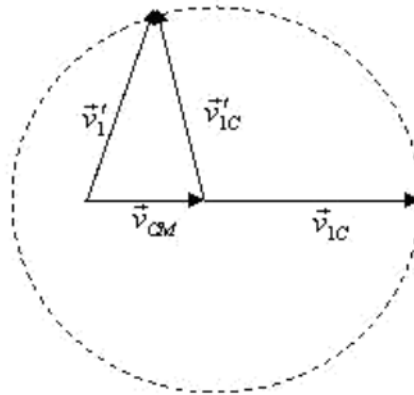
Deflection angle of particle 1 in the lab frame is maximum when velocities \vec{v}_1' and \vec{v}_{1C}' are perpendicular.

Figure 7

It is clear from figure 7 that the deflection angle of particle 1, when hitting another particle of smaller mass, increases as θ_{CM} increases from zero. It is maximum when the velocities \vec{v}_1' and \vec{v}_{1C}' are perpendicular. If \vec{v}_{1C}' rotates beyond this angle, deflection starts becoming smaller. Thus θ_{max} is given by the formula

$$\sin \theta_{\max} = \frac{v_{1C}'}{v_{CM}} = \frac{v_{1C}}{v_{CM}} = \frac{m_2}{m_1}$$

It is clear from the expression above that when a particle hits a lighter particle at rest, it is deflected by a small angle. This is reasonable as a light particle can hardly deflect a heavier particle. Thus the heavier particle keeps on moving forward even after the collision. On the other hand, there is no restriction on the scattering angle when a light particle hits a heavier particle at rest i.e. $m_1 < m_2$. In this case $v_{CM} < v_{1C}$ and therefore the graphical representation of different velocities is as shown in figure 8.



Deflection angle of particle 1 in the lab can take any value when $m_1 < m_2$.

Figure 8

It is clear from the figure that as θ_{CM} increases, so does θ_{lab} . In this situation, however, there is no restriction on the value that θ_{lab} can take as θ_{CM} sweeps angles from 0 to 2π .

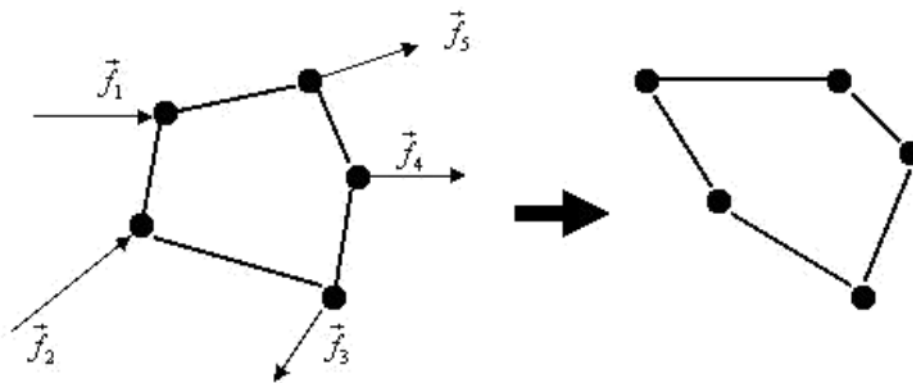
So far we have focused on elastic collisions only and could learn a great deal about them from conservation laws for momentum and energy. Such general conclusions are difficult to draw for inelastic collisions. As discussed in the beginning of this lecture, for inelastic collisions, we can definitely say that the maximum possible loss of energy is equal to the kinetic energy of particles in their CM frame. This would occur when the colliding particles get stuck together so that their kinetic energy after collision is zero in the CM frame. This concludes our lecture on collisions as analyzed using conservation laws.

Rotational dynamics I: Angular momentum

So far we have applied Newton's laws to point particles and the CM motion for a collection of particles. We are now going to look at what happens beyond the motion of the CM, which is described by the equation

$$\vec{F}_{total} = M_{total} \frac{d^2 \vec{R}_{CM}}{dt^2}$$

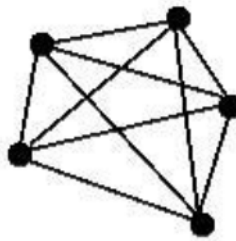
Let us see what else could happen to a body made up of a collection of particles where forces are applied at each point (figure 1). The particles are connected with flexible attachments shown as lines.



Distortion of a collection of particles connected with each other moving under force on each particle.

Figure 1

In the figure above, although the CM moves with $\vec{R}_{CM} = \frac{\vec{P}_{Net}}{M}$, the body itself could deform and change its orientation. Thus the distances between the particles and the angles between lines joining them would change. This is the most general motion that could take place. In the next few lectures we want to focus only one of the effects of the force applied. We are going to assume that a body only changes its orientation but does not deform. This is achieved by keeping the distance between any two particles of the body unchanged. Such a body is known as a *rigid body*. Thus in the example above, if we connect all the particles with each other by rods of fixed length, the body will become rigid. This is shown in figure 2.

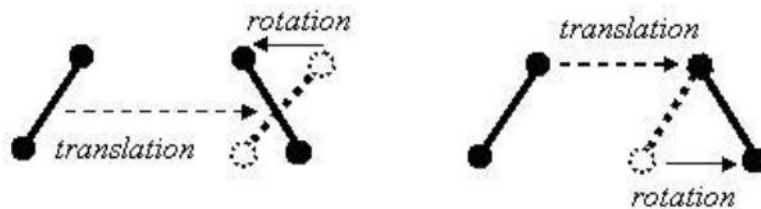


A body is rigid if the distance between any of its two particles remains unchanged.

Figure 2

The only possible motion of such a body is a translation plus a change in its orientation. The simplest example of a rigid body is two masses attached at the ends of a rod of fixed length. On the other hand, a tin-can partially filled with sand is not a rigid body since the distance between two particles keeps on changing with the motion of the can.

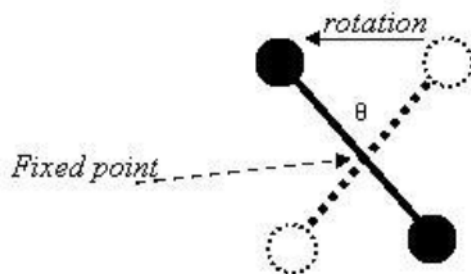
As stated above, the most general motion of a rigid body is its translation plus its change of orientation. The latter is equivalent to a rotation about a point. The beauty of this decomposition is that to get the final position of the body, we can translate any point in the body and then rotate the body about that point. Irrespective of which point we choose, the sense and the angle of rotation is always the same. Usually this point is taken to be the CM for reasons that will become clear later lectures. This general motion is shown below in figure 3, giving two possible ways of translating and rotating the body.



The most general motion of a rigid body is its translation plus a rotation. Two such possible ways giving the same final orientation are shown. In both cases the sense and angle of rotation are the same

Figure 3

You see that in figure 3 the rigid body has translated and also rotated. On the other hand, if we keep one of the points on the body fixed the only thing the body can do is to change its orientation (see figure 4). Thus with a point fixed, the only possible motion of a rigid body is a rotation.



The only possible motion of a rigid body with one of its points fixed is a rotation.

Figure 4

A question that arises now is how many variables do we need to specify the general motion of a rigid body. It requires three variables - x , y and z coordinates of the point that is translated - to describe the translation, and three more - angle of rotation about each axis - to represent the rotation. You can see that in general a rigid body would require six variables to describe its

motion. However, if one of its points is fixed, three variables are sufficient to specify its rotation. So we conclude *a rigid body needs six parameters to describe its motion.*

For simplicity, in the beginning we are going to focus on rigid moving with its one point fixed. Thus it will change only by changing its orientation. We will further simplify the problem by considering rotation about an axis fixed in space. In the next step, we will allow the axis to translate but without changing its orientation. Finally we will also let the orientation of the axis change. Thus we will increase the complexity of the problem gradually.

Dynamics of rigid body: The dynamics of a rigid body is best described by considering its angular momentum. You can think of angular momentum as the rotational counter part of linear momentum. This quantity is central to describing rotational motion of a rigid body. So let us first spend some time in understanding this quantity. Although we are introducing angular momentum here in the context of rigid bodies, the treatment below is quite general.

For a single particle moving with linear momentum \vec{p} at a distance \vec{r} from the origin the angular momentum \vec{L} is defined as

$$\vec{L} = \vec{r} \times \vec{p}$$

You can immediately see that it is an origin-dependent quantity. If we calculate it with respect to some other point, it will come out to be different. If a particle of mass m is moving in a plane then using the polar coordinates for it, it is easily shown that its angular momentum is

$\vec{L} = mr^2 \dot{\phi} \hat{z}$. Let us now find out what is the rate of change of angular moment? It is calculated below.

$$\begin{aligned} \frac{d\vec{L}}{dt} &= \frac{d}{dt}(\vec{r} \times \vec{p}) \\ &= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} \end{aligned}$$

With $\frac{d\vec{r}}{dt} = \vec{v}$, $\vec{p} = m\vec{v}$ and $\frac{d\vec{p}}{dt} = \vec{F}$, where \vec{F} is the force on the particle, the equation above is simplified to

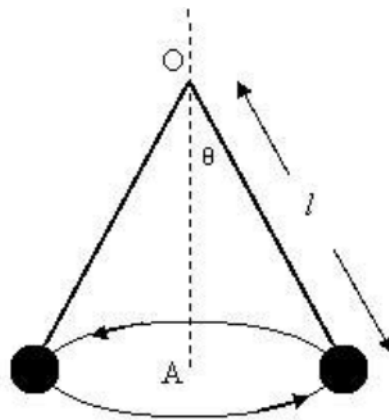
$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau} \text{ (torque)}$$

Thus rate of change of angular momentum is equal to the torque applied on the body. From the equation above, the law of *conservation of angular momentum* follows immediately: If the applied torque $\vec{\tau} = 0$ the angular momentum \vec{L} does not change, i.e. it is a constant. The equation

$$\frac{d\vec{L}}{dt} = \vec{\tau}$$

is the angular momentum equivalent of Newton's IInd law. Let us now illustrate the ideas presented so far with the example of a conical pendulum.

Example 1: A conical pendulum is like the regular pendulum with a light (mass $m = 0$) rigid rod carrying a bob of mass m at one of its ends. The other end is fixed and the bob moves in a circle with speed v (see figure 6). We wish to calculate the tension in the rod and the angle θ it makes from the vertical by applying the angular momentum-torque equation.



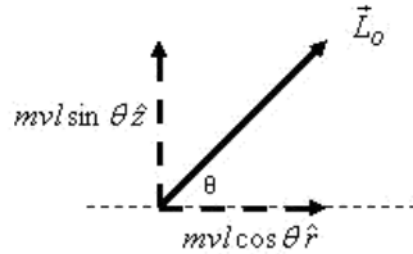
A conical pendulum

Figure 6

Let us first calculate angular momentum about point O . We will use cylindrical coordinates because of the symmetry of the problem. With respect to O

$$\begin{aligned}\vec{r} &= l \sin \theta \hat{r} - l \cos \theta \hat{z} \quad \text{and} \quad \vec{v} = v \hat{\phi} \\ \vec{L}_O &= m \vec{r} \times \vec{v} \\ &= m(l \sin \theta \hat{r} - l \cos \theta \hat{z}) \times v \hat{\phi} \\ &= mvl \sin \theta \hat{z} + mvl \cos \theta \hat{r}\end{aligned}$$

The \vec{L}_O vector looks as shown in figure 7, when the bob of the pendulum is in the paper plane.



The angular momentum and its components for a conical pendulum

Figure 7

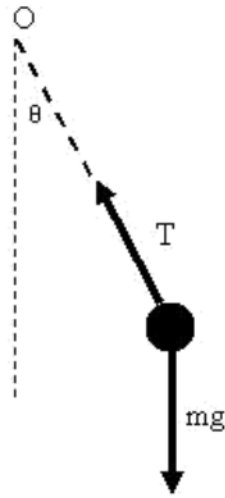
So the angular momentum \vec{L} is perpendicular to the rod (take the dot product with $\vec{r} = l \sin \theta \hat{r} - l \cos \theta \hat{z}$ for mass m and see for yourself) and as the particle rotates the horizontal component of \vec{L} are rotates with it and the vertical component remains a constant. Let us now apply the equation

$$\frac{d\vec{L}_O}{dt} = \vec{\tau}_O$$

We have

$$\begin{aligned} \frac{d\vec{L}_O}{dt} &= \frac{d}{dt} (mvl \sin \theta \hat{z} + mvl \cos \theta \hat{r}) \\ &= mvl \cos \theta \frac{d}{dt} \hat{r} \\ &= mvl \cos \theta \dot{\phi} \hat{\phi} \\ &= mvl \cos \theta \times \frac{v}{l \sin \theta} \hat{\phi} \\ &= mv^2 \cot \theta \hat{\phi} \end{aligned}$$

We now calculate the torque acting on the pendulum. There are two forces, the tension \vec{T} and the weight $\vec{W} = -mg\hat{k}$, acting on the particle as shown in figure 8.



Forces acting on the mass m of the conical pendulum

Figure 8

But \vec{T} passes through O and does not give any torque. Thus

$$\begin{aligned}\vec{\tau}_O &= (l \sin \theta \hat{r} - l \cos \theta \hat{z}) \times mg(-\hat{z}) \\ &= mgl \sin \theta \hat{\phi}\end{aligned}$$

Substituting these in the angular momentum-torque equation $\frac{d\vec{L}_O}{dt} = \vec{\tau}_O$ then gives

$$\frac{v^2}{gl} = \tan \theta \sin \theta$$

The angular momentum-torque equation therefore gives us the angle θ that the pendulum makes with the vertical. How do we find the tension T ? On the other hand, applying Newton's second Law we get

$$T \cos \theta = mg \quad \text{and} \quad T \sin \theta = \frac{mv^2}{l \sin \theta}$$

giving

$$T = \frac{mg}{\cos \theta} \quad \text{and} \quad \tan \theta \sin \theta = \frac{v^2}{gl}$$

These equations give us both T and θ , but the equation $\frac{d\vec{L}_O}{dt} = \vec{\tau}_O$ gives only the angular relationship. Does this mean that the angular-momentum torque equation is not equivalent to Newton's second law? The answer is that it is. It so happens that in applying the equation about O , when cross products $\vec{r} \times \vec{p}$ and $\vec{r} \times \vec{F}$ are taken, some components of the force do not contribute to the torque and drop out of the equation. For example in this case $\vec{r} \times \vec{T}$ becomes zero. To get full solution, therefore, we now apply $\frac{d\vec{L}}{dt} = \vec{\tau}$ about point A. Taking A as the origin we have

$$\begin{aligned}\vec{r} &= l \sin \theta \hat{r} \quad \text{and} \quad \vec{v} = v \hat{\phi} \\ \vec{L}_A &= m l \sin \theta \hat{r} \times v \hat{\phi} \\ &= m v l \sin \theta \hat{z}\end{aligned}$$

Since all the quantities in \vec{L}_A are constants, we have

$$\vec{\tau}_A = \frac{d\vec{L}_A}{dt} = 0$$

Let us calculate the torque $\vec{\tau}_A$ about A. With A as the origin, the forces are given as

$$\vec{T} = -T \sin \theta \hat{r} + T \cos \theta \hat{z} \quad \text{and} \quad m\vec{g} = -mg \hat{z}$$

Therefore

$$\begin{aligned}\vec{\tau}_A &= l \sin \theta \hat{r} \times \{-T \sin \theta \hat{r} + (T \cos \theta - mg) \hat{z}\} = 0 \\ \Rightarrow (T \cos \theta - mg) l \sin \theta \hat{\phi} &= 0\end{aligned}$$

which gives

$$T = \frac{mg}{\cos \theta}$$

Thus applying $\frac{d\vec{L}}{dt} = \vec{\tau}$ about two different points gives exactly the same solution as that obtained from $\frac{d\vec{p}}{dt} = \vec{F}$. Thus the two ways of solving the problem are equivalent. Through this example I

have shown you (a) the origin dependence of \vec{L} and $\vec{\tau}$, and (b) equivalence of $\frac{d\vec{L}}{dt} = \vec{\tau}$ and $\frac{d\vec{p}}{dt} = \vec{F}$.

Let me now illustrate conservation of angular momentum by a well known example: that of Kepler's Law of equal area concept in equal time. Accordingly, when planets are going around the sun, the rate at which their position vector from the sun sweeps the area is a constant. Recall from the lecture on polar coordinates that for a particle moving under a radial force, we had

obtained that $r^2\dot{\phi}$ is a constant. This is nothing but two times the rate of area sweep by the radius vector. We now want to get this law from the conservation of angular momentum.

For a planet, we know that the force is in radial direction. So that the torque

$$\vec{\tau} = \vec{r} \times \vec{F} = 0$$

Thus

$$\frac{d\vec{L}}{dt} = \vec{\tau} = 0 \text{ or } \vec{L} = \text{constant}$$

Since $\vec{L} = m\vec{r} \times \vec{v} = m\vec{r} \times (\dot{r}\hat{r} + r\dot{\phi}\hat{\phi}) = mr^2\dot{\phi}\hat{z}$, its constancy means

$$r^2\dot{\phi} = \text{constant}$$

which is Kepler's second law.

After this initial demonstration of $\frac{d\vec{L}}{dt} = \vec{\tau}$ with a single particle, we move on to a system of many particles. It is really a system of many particles that we are dealing with in rigid-body dynamics.

Angular Momentum of a collection of particles: If there are many particles then the total angular momentum \vec{L} about a point O is the sum of individual angular momenta of each particle about O . Thus

$$\begin{aligned} \vec{L} &= \sum_i \vec{L}_i \\ &= \sum_i m_i \vec{r}_i \times \vec{p}_i \end{aligned}$$

As for the angular momentum of a single particle, the angular momentum of a many-particle system is also origin-dependent. (**Question:** Under what conditions will the angular momentum be independent of the origin?)

Now recall that the kinetic energy for a collection of particles is the sum of the kinetic energy of their centre of mass (CM) and the kinetic energy of particles with respect to the CM. Interestingly the angular momentum of a many-particle system can be expressed in the same manner. Thus the total angular momentum of a collection of particles is equal to the angular momentum of the CM plus the angular momentum of particles about the CM. Let us now prove it. To do so express the position vector and the velocity of a particle as

$$\vec{r}_i = \vec{r}_{CM} + \vec{r}_{iC} \quad \text{and} \quad \vec{v}_i = \vec{v}_{CM} + \vec{v}_{iC}$$

where \vec{r}_{CM} and \vec{v}_{CM} refer to the position and velocity of the CM and \vec{r}_{iC} and \vec{v}_{iC} the position and velocity of i^{th} particle with respect to the CM. Now the total angular momentum can be written as

$$\begin{aligned} \vec{L} &= \sum_i m_i (\vec{r}_{CM} + \vec{r}_{iC}) \times (\vec{v}_{CM} + \vec{v}_{iC}) \\ &= \left(\sum_i m_i \right) \vec{r}_{CM} \times \vec{v}_{CM} + \vec{r}_{CM} \times \left(\sum_i m_i \vec{v}_{iC} \right) \\ &\quad + \left(\sum_i m_i \vec{r}_{iC} \right) \times \vec{v}_{CM} + \sum_i m_i \vec{r}_{iC} \times \vec{v}_{iC} \end{aligned}$$

$$\left(\sum_i m_i \vec{r}_{iC} \right) = 0 \quad \text{and} \quad \left(\sum_i m_i \vec{v}_{iC} \right) = 0$$

However, by definition of the CM, . Therefore the second and the last term in the expression above do not contribute. The remaining terms are written as

$$\begin{aligned} \vec{L} &= \left(\sum_i m_i \right) \vec{r}_{CM} \times \vec{v}_{CM} + \sum_i m_i \vec{r}_{iC} \times \vec{v}_{iC} \\ &= M \vec{r}_{CM} \times \vec{v}_{CM} + \sum_i m_i \vec{r}_{iC} \times \vec{v}_{iC} \\ &= \vec{L}_{CM} + \vec{L}_{aboutCM} \end{aligned}$$

where M is the total mass of the system. This is a remarkable result, and as we will see, facilitates calculations involving rigid-body dynamics a lot. Keep in mind though that this result is true only with for the CM. For an arbitrary point O' in the body, we cannot write

$$\vec{L} = \vec{L}_{O'} + \vec{L}_{aboutO'}$$

because $\vec{L} = \vec{L}_{CM} + \vec{L}_{about CM}$ depends explicitly on the definition of the CM. We will later use this fact to obtain the parallel axis theorem that you may have learnt in your previous classes. The theorem is similar to the transfer theorem of the second-moment of an area.

The relationship $\vec{L} = \vec{L}_{CM} + \vec{L}_{about CM}$ also tells us that if the total momentum of a system of particles is zero, its angular momentum will be independent of the origin. I leave the simple proof for you to work out.

Example: Take a bicycle wheel of radius R rolling along the ground and assume all its mass M is concentrated along the rim. If it is rolling without slipping then its motion is as follows: its CM moves with speed V along a straight line and the wheel rotates about the CM with angular speed $\omega = \frac{V}{R}$ so that the point on ground is at rest. We want to find its angular momentum in a frame stuck to the ground such that the wheel is moving along its x-axis see figure 9).

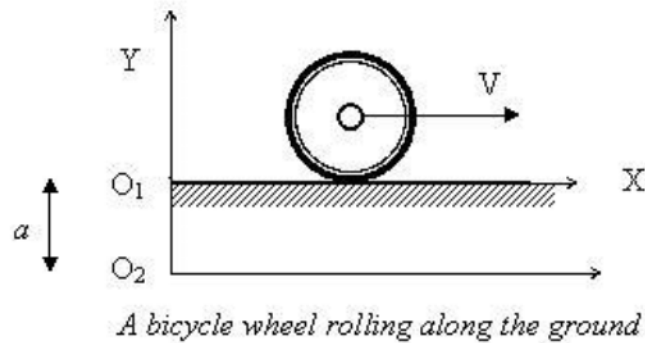


Figure 9

The angular momentum of the wheel about its CM is given as

$$\begin{aligned}\vec{L}_{about CM} &= \left(\sum_i m_i \right) RV \\ &= MRV\end{aligned}$$

So angular momentum about the origin O_1 (see figure 9) would be

$$\begin{aligned}&= MRV + \text{angular momentum of CM about } O_1 \\ &= MRV + MRV \\ &= 2MRV\end{aligned}$$

On the other hand, if we were to calculate the angular momentum about O_2 (see figure 9) it would come out to be

$$\begin{aligned}
&= MRV + \text{angular momentum of CM about } O_2 \\
&= MRV + M(R+a)V \\
&= 2MRV + MRa
\end{aligned}$$

Notice that in both the cases we have added the angular momentum of the CM and that about the CM. It is because their directions come out to be the same (negative z direction). One must be careful about these things because angular momentum is a vector quantity. Having introduced you to the concept of angular momentum, I now discuss about the rate of its change for a many-particle system where the particles are interacting with each other also.

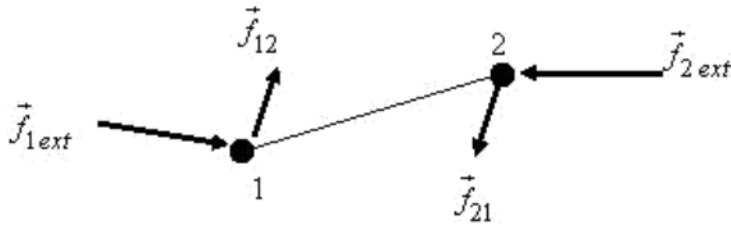
Dynamic of a rigid body; $\frac{d\vec{L}}{dt}$ and conservation of angular momentum: Let us now look at $\frac{d\vec{L}}{dt}$ in the case of a collection of particles which are interacting with each other and are also being acted upon by external forces.

$$\begin{aligned}
\frac{d\vec{L}}{dt} &= \frac{d}{dt} \sum_i m_i \vec{r}_i \times \vec{v}_i \\
&= \sum_i m_i \vec{r}_i \times \frac{d\vec{v}_i}{dt}
\end{aligned}$$

But $m_i \frac{d\vec{v}_i}{dt} = \vec{f}_i$ (\vec{f}_i is the total force, i.e. the sum of external and internal forces on the particle). This gives

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{f}_i$$

Before simplifying this equation in terms of the external torque, let us see where does this equation lead us for a two particle system shown in figure 10?



A two particle system with particles interacting with each other and are also being acted upon by external forces

Figure 10

The two particles 1 and 2 shown in figure 10 are external forces \vec{f}_{1ext} and \vec{f}_{2ext} , respectively. They also interact with each other with particle 2 applying a force \vec{f}_{12} on particle 1 and particle 1 applying a force \vec{f}_{21} on particle 2. We assume the forces to be following Newton's IIIrd law so that $\vec{f}_{21} = -\vec{f}_{12}$. Now the rate of change for this system can be written as

$$\begin{aligned}\frac{d\vec{L}}{dt} &= \vec{r}_1 \times (\vec{f}_{1ext} + \vec{f}_{12}) + \vec{r}_2 \times (\vec{f}_{2ext} + \vec{f}_{21}) \\ &= \vec{r}_1 \times \vec{f}_{1ext} + \vec{r}_2 \times \vec{f}_{2ext} + \vec{r}_1 \times \vec{f}_{12} + \vec{r}_2 \times \vec{f}_{21} \\ &= \vec{r}_1 \times \vec{f}_{1ext} + \vec{r}_2 \times \vec{f}_{2ext} + (\vec{r}_1 - \vec{r}_2) \times \vec{f}_{12} \\ &= \vec{\tau}_{ext} + (\vec{r}_1 - \vec{r}_2) \times \vec{f}_{12}\end{aligned}$$

Thus the rate of change of angular momentum is equal to only the external torque if $(\vec{r}_1 - \vec{r}_2) \times \vec{f}_{12} = 0$ or $\vec{f}_{12} \parallel (\vec{r}_1 - \vec{r}_2)$, i.e. the force between the particles is along the line joining them. At this point I would like you to recall that in the case of linear momentum, the rate of change on linear momentum equals the total external force, i.e. $\frac{d\vec{p}}{dt} = \vec{F}_{ext}$, only if $\vec{f}_{12} = -\vec{f}_{21}$. For

angular momentum to satisfy $\frac{d\vec{L}}{dt} = \vec{\tau}_{ext}$, the additional condition of $\vec{f}_{12} \parallel (\vec{r}_1 - \vec{r}_2)$ is also needed. Fortunately for most of the mechanical applications this is true. Let us now generalize this to the case of a many-particle system. For such a system

$$\begin{aligned}\frac{d\vec{L}}{dt} &= \sum_i \vec{r}_i \times \vec{f}_i \\ &= \sum_i \vec{r}_i \times \vec{f}_{ext} + \sum_i \sum_{j, j \neq i} \vec{r}_i \times \vec{f}_{ij}\end{aligned}$$

Recall the trick used in the case of linear momentum that

$$\sum_{i,j} \vec{r}_i \times \vec{f}_{ij} = \sum_{i,j} \vec{r}_j \times \vec{f}_{ji}$$

so that

$$\begin{aligned} \sum_{i,j} \vec{r}_i \times \vec{f}_{ij} &= \frac{1}{2} \sum_{i,j} [\vec{r}_i \times \vec{f}_{ij} + \vec{r}_j \times \vec{f}_{ji}] \\ &= \frac{1}{2} \sum_{i,j} [(\vec{r}_i - \vec{r}_j) \times \vec{f}_{ij}] \quad (\vec{f}_{ji} = -\vec{f}_{ij} \text{ by Newton's III Law}) \\ &= 0 \text{ if } \vec{f}_{ij} \parallel (\vec{r}_i - \vec{r}_j) \end{aligned}$$

Under these conditions, i.e. *if the force between the particles is along the line joining them*, we get

$$\begin{aligned} \frac{d\vec{L}}{dt} &= \sum_i \vec{r}_i \times \vec{f}_{i \text{ ext}} \\ &= \vec{\tau}_{\text{ext}} \end{aligned}$$

Thus if $\vec{\tau}_{\text{ext}} = 0$ then $\frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L} = \text{constant}$. Thus is the law of *conservation of total angular momentum*. In the next lecture we will do a few example of its application.

We now conclude this lecture by listing the following points that we have learnt:

1. A rigid body needs six parameters to describe its general motion; three for translation and three for rotation,
2. Dynamics of rigid body is governed by its angular momentum,
3. The angular momentum satisfies the equation

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\text{ext}}$$

under the condition that the internal forces satisfy Newton's IIIrd law and an additional condition that $\vec{f}_{ij} \parallel (\vec{r}_i - \vec{r}_j)$

$$4. \vec{L} = \vec{L}_{CM} + \vec{L}_{about CM}$$

Rotational dynamics II: Rotation about a fixed axis

We saw in the previous lecture on rigid bodies that a rigid body in general requires six parameters to describe its motion, and the dynamic of a rigid body is determined through its

angular momentum \vec{L} that satisfies the equation $\frac{d\vec{L}}{dt} = \vec{\tau}_{ext}$, where $\vec{\tau}_{ext}$ is the applied torque on the body. Further, $\vec{\tau}_{ext} = 0$ means that \vec{L} is a constant.

In this lecture I start with an example of the conservation of angular momentum involving two particles. I again show that a direct application of Newton's laws and a solution through the conservation of angular momentum give the same answer.

Example 1: There is a rigid massless rod of length b held at point O carrying a mass m_2 at its other end. Let the y -coordinate of m_2 be a . Another mass m_1 comes parallel to the x -axis and hits m_2 and the two masses get stuck together (see figure 1). Question is at what speed will the rod rotate?

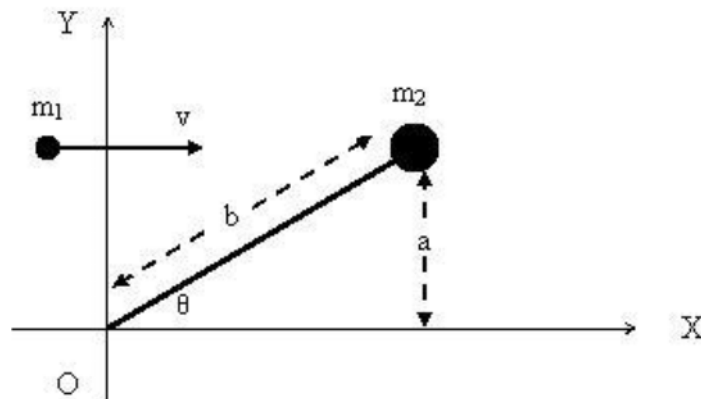


Figure 1

Let us apply the conservation of angular momentum to the system of two masses about point O . This is because the only external force acts at O so the torque about O is zero and therefore the angular momentum about O is conserved. Since the particles are moving in the xy -plane, their angular momentum is going to be in the z direction. So we write the unit vector explicitly and work in terms of numbers (both positive and negative) only. Assume that the angular velocity of the rod after the mass m_1 gets stuck with it is ω . To apply angular momentum conservation we calculate the angular momentum of the system before and after collision and equate them.

Initial angular momentum about O = $m_1 v a$

Final angular momentum = $(m_1 + m_2) b^2 \omega$

Equating the two gives

$$\omega = \frac{m_1 v a}{(m_1 + m_2) b^2}$$

Let us now see if the conventional force analysis also gives the same answer. The incoming mass m_1 comes in with momentum $m_1 v$. Now after m_2 is hit, it cannot have any movement parallel to the rod because the rod is rigid, i.e. the rod is capable of generating enough tension (impulse) in it to make the component of momentum parallel to the rod zero. On the other hand, there is no force perpendicular to the rod so the momentum component p_{\perp} in that direction remains unchanged after the hit. Now

$$p_{\perp} = m_1 v \sin \theta = \frac{m_1 v a}{b}$$

After the masses get stuck together, p_{\perp} remains the same. Thus the new speed v' acquired by the masses will be such that

$$(m_1 + m_2) v' = \frac{m_1 v a}{b} \quad \text{or} \quad v' = \frac{m_1 v a}{(m_1 + m_2) b}$$

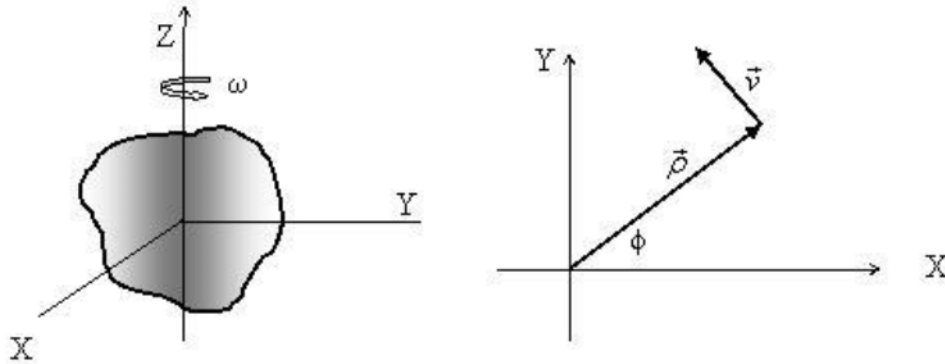
This gives

$$\omega = \frac{v'}{b} = \frac{m_1 v a}{(m_1 + m_2) b^2}$$

which is the same as obtained by angular momentum conservation. Thus again showing the equivalence of the two methods.

With all this preparation, let us now start with the simplest motion of a rigid body that is the rotation of a rigid body about an axis fixed in space. So the axis is neither translating nor rotating. Without any loss of generality, let us call this axis the z-axis. In this case the body has only one degree of freedom and the only variable that we need to describe the motion of the body is the angle of rotation about the axis. Further, the only relevant component of angular momentum in this case is the component along the z-axis. Note that there may be other components of angular momentum but their change is accounted for by torques applied on the axis to keep it fixed in space. Calculation of such torques will be discussed in later lectures. Suffices here is to say that these torques arise out of the constraint forces that enforce the constraint of the axis being fixed in space.

Shown in figure 2 is a rigid body rotating about the z-axis with an angular speed ω . Also shown there is the position and velocity vector of one of its constituent particles of mass m_i in a plane perpendicular to the rotation axis. We wish to calculate the z component of the angular momentum.



A rigid body rotating about the z-axis (left) and the position and velocity vectors of a point in it (right) shown in the plane perpendicular to the z-axis

Figure 2

The z component will be given as

$$L_z = \sum_i m_i (\vec{r}_i \times \vec{v}_i)_z = \sum_i m_i (x_i v_{iy} - y_i v_{ix})$$

For a particle at distance ρ_i from the z-axis and its radius vector making an angle ϕ_i from the x-axis

$$\begin{aligned} x_i &= \rho_i \cos \phi_i & v_{ix} &= -\omega \rho_i \sin \phi_i \\ y_i &= \rho_i \sin \phi_i & v_{iy} &= \omega \rho_i \cos \phi_i \end{aligned}$$

so that

$$L_z = \left(\sum_i m_i \rho_i^2 \right) \omega$$

Calling $I_z = \left(\sum_i m_i \rho_i^2 \right)$ the moment of inertial about the rotation axis, we can write

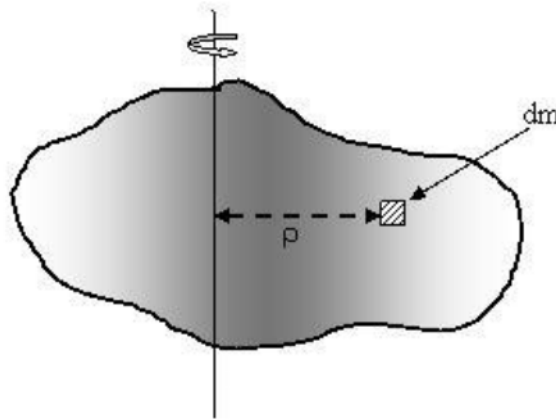
$$L_z = I_z \omega$$

Depending on the direction of ω , angular momentum about an axis could have negative or positive values because it is a vector quantity. The convention we take is the right-hand convention; Let the thumb of one's right hand point in the positive z direction; if the rotation of the body is in the same (opposite) direction as the fingers, ω is positive (negative).

Having defined the moment of inertia about an axis, we make a few comments on it. First thing we notice about it is that it depends on the perpendicular distance of point masses from the axis of rotation. So no matter where we take the origin of the coordinate system, the moment of inertia of a rigid body about an axis is always going to be the same. Secondly, for continuously distribute mass moment of inertia is calculated as the integral

$$I_z = \int \rho^2 dm$$

where ρ is the perpendicular distance of a small mass element dm taken in the body (see figure 3).



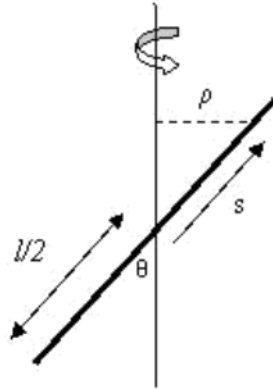
A small mass element dm in a rigid body at a distance ρ from the axis of rotation

Figure 3

Finally, for planar objects the moment of inertia is the same as the second moment of an area except that the area is replaced by the mass.

We now calculate moment of inertia of some objects.

A rod at an angle from the axis of rotation passing through its centre: This is shown in figure 4. The length of the rod is l and its mass m . It is at an angle θ from the axis of rotation.



A rod at an angle from the axis of rotation

Figure 4

We take a small mass element of length ds at a distance s from the origin. It is at a distance $\rho = s \sin \theta$ from the axis of rotation. Then

$$\begin{aligned} I &= \int \rho^2 dm \\ &= \int_{-l/2}^{l/2} s^2 \sin^2 \theta \frac{m}{l} ds \\ &= \frac{ml^2}{12} \sin^2 \theta \end{aligned}$$

Thus for a rod rotating about its perpendicular passing through its centre is $\frac{ml^2}{12}$.

Exercise: Calculate the moment of inertia of a disc rotating about an axis passing through its centre and perpendicular to it.

Moment of inertia of disc about one of its diameters: Shown in figure 5 is a disc of mass M and radius R rotating about its diameter which lies on the y-axis.

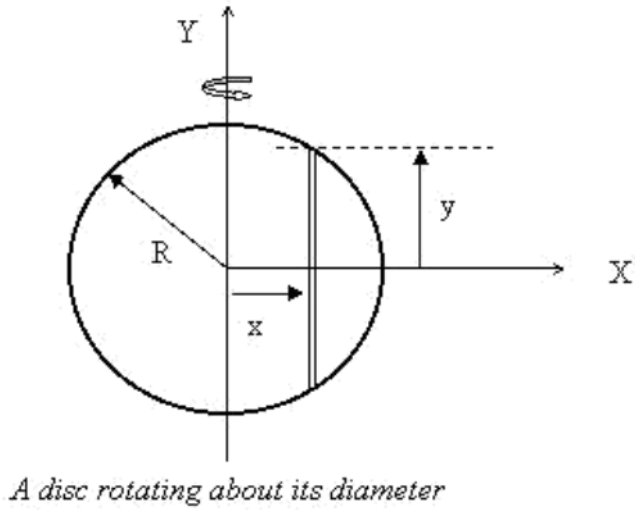


Figure 5

To calculate the moment of inertia I take a strip of length width dx at distance x from the y -axis,

the axis of rotation. Its mass is $dm = \frac{M}{\pi R^2} 2y dx$ (see figure 5). Thus

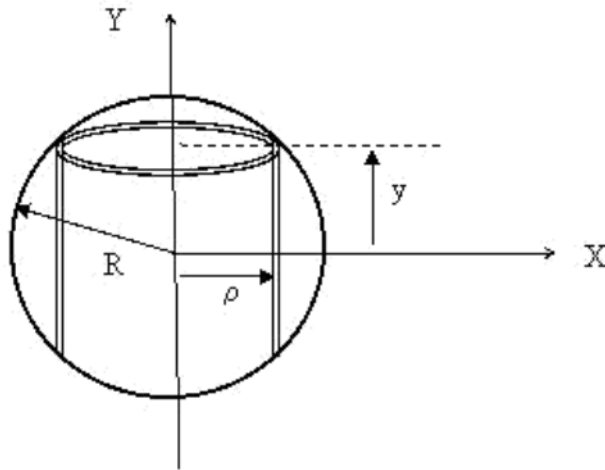
$$\begin{aligned}
 I &= \int x^2 dm \\
 &= \int_{-R}^R x^2 \frac{M}{\pi R^2} 2y dx \\
 &= \frac{2M}{\pi R^2} \int_{-R}^R x^2 \sqrt{R^2 - x^2} dx
 \end{aligned}$$

The integration can be carried out easily by substituting $x = R \cos \theta$ and gives

$$I = \frac{MR^2}{4}$$

Moment of inertia of a sphere about one of its diameters: A sphere of mass M and radius R is shown in figure 6. To calculate its moment of inertia, we take a cylindrical shell of radius ρ and thickness $d\rho$ (see figure 6). The mass of this shell is given by

$$\begin{aligned}
 dm &= \frac{M}{4\pi R^3/3} \times 2\pi \rho d\rho \times 2y \\
 &= \frac{3M}{R^3} \sqrt{R^2 - \rho^2} \rho d\rho
 \end{aligned}$$



A sphere rotating about its diameter

Figure 6

Therefore the moment of inertia is

$$I = \int \rho^2 dm = \frac{3M}{R^3} \int_0^R \rho^3 \sqrt{R^2 - \rho^2} d\rho$$

By substituting $\rho = R \cos \theta$, this is an easy integral to perform and gives the result

$$I = \frac{2}{5} MR^2$$

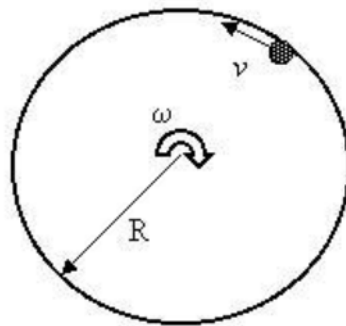
Let us now recapture what we have done so far. We have looked at the angular momentum of a body rotating about a fixed axis. We find that angular momentum L_z about an axis (denoted as the z-axis) is given as $L_z = I_z \omega$ and, depending upon the sense of rotation, can take positive as well as negative values. We have also calculated I_z for some standard objects about an axis. We now go on to study the equation of motion satisfied by L_z . The equation satisfied by L_z is

$$\frac{dL_z}{dt} = \tau_z$$

where τ_z is the component of the *external* torque along the axis of rotation. If the external torque is zero, the angular momentum is conserved. You can observe the effect of conservation of angular momentum easily at home.

Sit on a revolving chair holding a brick (or something similar) in each of your hands and keep your arms stretched. Start revolving the chair and then pull your arms in. You will observe that you start revolving much faster. This happens because when you pull the arms in, the masses that you are holding come closer to the axis of rotation resulting in a reduction in the value of the moment of inertia. However, since there is no external torque on the system, the angular momentum cannot change. Thus if the moment of inertia decreases, the angular speed must increase in order to keep $L = I\omega$ constant. This is precisely what you observe. You should also repeat the experiment holding different weights. When do you observe the rotational speed to increase the largest? Let us now solve an example of applying the angular momentum conservation principle.

Example: A man starts walking on the edge of a circular platform with a speed v with respect to the platform (see figure 7). The platform is free to rotate. What is the rotational speed of the platform? Mass of the platform is M , its radius is R and the mass of the man is m .



A man of mass m walks on the edge of a circular platform of mass M and radius R . the platform in turn starts rotating.

Figure 7

Since there is no external torque, the angular momentum of the system about the axis of rotation must be conserved. Thus as the man starts walking, the platform starts rotating the other way. Since the speed of man with respect to the platform is v , his speed in the ground frame would be $(v - \omega R)$. Thus the angular momentum of the man is

$$mR(v - \omega R)$$

At the same time, the angular momentum of the platform is

$$-\frac{1}{2}MR^2\omega$$

where the minus sign shows that the angular momentum of the platform is in the direction opposite to that of the man's angular momentum. By conservation of angular momentum

$$mR(v - \omega R) - \frac{1}{2}MR^2\omega = 0$$

which gives

$$\omega = \frac{v/R}{\left(1 + \frac{M}{2m}\right)}$$

Having learnt about the angular momentum, its equation of motion and the conservation of angular momentum for rotations about a fixed axis, we now discuss the kinetic energy and the work-energy theorem for a rigid body rotating with angular speed ω about a fixed axis.

Kinetic energy and work-energy theorem for a rigid-body rotating about a fixed axis: The kinetic energy of a rigid body rotating with angular speed ω is obtained by calculating the energy of small mass element in the body and adding it up. This mass element is rotating in a plane perpendicular to the axis of rotation. This gives (using the notation of figure 2)

$$\begin{aligned} K.E. &= \frac{1}{2} \sum_i m_i v_i^2 \\ &= \frac{1}{2} \left(\sum_i m_i \rho_i^2 \right) \omega^2 \\ &= \frac{1}{2} I \omega^2 \end{aligned}$$

The corresponding work-energy theorem for the motion considered here is that the change in kinetic energy is equal to the work done on the body. Let us first calculate the work done on a body, which can only rotate about an axis, when an external force is applied on it. To do this, I would first like you to prove a result (look at figure 2 for reference): when a body rotates by an angle $\Delta\theta$ about an axis in the unit vector direction \hat{n} , the corresponding change in position of a particle in the body at position vector \vec{r}_i is

$$\Delta \vec{r}_i = (\hat{n} \times \vec{r}_i) \Delta \theta$$

The total work done on the body by a net external force composed of forces \vec{f}_i acting at each point is

$$\Delta W = \sum_i \vec{f}_i \cdot (\hat{n} \times \vec{r}_i) \Delta \theta$$

By using $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{A} \times \vec{C})$, we can write the work done as

$$\begin{aligned} \Delta W &= \sum_i \hat{n} \cdot (\vec{r}_i \times \vec{f}_i) \Delta \theta \\ &= \hat{n} \cdot \left(\sum_i (\vec{r}_i \times \vec{f}_i) \right) \Delta \theta \\ &= \hat{n} \cdot \vec{\tau} \Delta \theta \\ &= \tau_z \Delta \theta \end{aligned}$$

where τ_z is the component of the external torque along the axis of rotation. Thus the total work done is

$$W = \int \tau_z d\theta$$

Now the work energy theorem can be expressed as follows:

$$\Delta K.E. = \int \tau_z d\theta$$

This pretty much concludes what all I have to say about the rotations about a fixed axis. One question that may be asked at this point is: Why is it what describing dynamics in term of angular momentum, torque etcetera rather than momentum and force is more useful in discussing rotational motion. This is because in rotational motion, force, momenta etcetera are distributed and taking their moments by considering the angular momenta and torques automatically takes care of this distribution. We conclude this lecture by drawing a comparison between linear and rotational motion about a fixed axis.

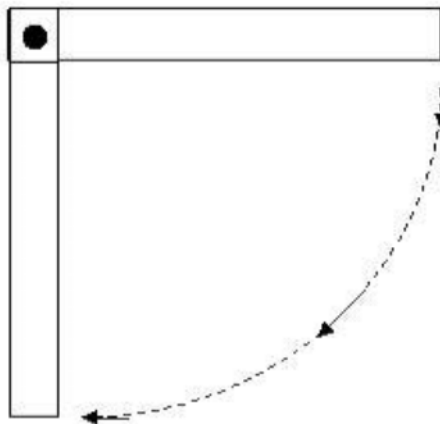
Linear motion	Rotational motion about a fixed axis
Momentum p	Angular momentum L
$\frac{dp}{dt} = F$	$\frac{dL}{dt} = \tau$
Impulse $\int F dt = \Delta p$	Impulse $\int \tau dt = \Delta L$
$K.E. = \frac{1}{2} m v^2$	$K.E. = \frac{1}{2} I \omega^2$
$\Delta K.E. = \int F dx$	$\Delta K.E. = \int \tau d\theta$

This correspondence will help in understanding and getting relationships to solve most of the problems involving rotations about a fixed axis, particularly if you have solved many problems involving linear momentum.

Rigid body dynamics III: Rotation and Translation

We have seen in the past two lectures how do we go about solving the rigid body dynamics problem by considering the rate of change of angular momentum. In the previous lecture, we concentrated on rotation about a fixed axis and solved problems involving conservation of angular momentum about that axis. In this lecture we consider what happens where an external torque is applied and also when the axis is allowed to translate parallel to itself.

Let us first take the case when the axis is stationary and a torque is applied. Take for instance your pen or a scale and hold it lightly at one of its ends so as to pivot it there. Raise the other end so that the scale is horizontal and then leave it. You will see that the scale swings down. I would like to calculate the speed of its CM when the scale is vertical after being released from horizontal position (see figure 1). Assume that there is no loss due to friction. In this case I will solve this problem in two ways and also comment on a wrong way.



A scale pivoted at one end and released from horizontal position swings down

Figure 1

I take the mass of the scale to be m and its length l . Then its moment of inertia about one of its

ends is $I = \frac{ml^2}{3}$.

I first solve the problem using energy conservation. Since there is no loss due to friction the total mechanical energy is conserved. Therefore the total mechanical energy is conserved. Let us take the potential energy to be zero when the scale is horizontal. Since the scale starts with zero initial angular speed, its total mechanical energy is zero. When the scale reaches the vertical position, its CM has moved down by a distance $l/2$ so its potential energy is $-mgl/2$. If its angular speed at that position is ω , then by conservation of energy

$$\frac{1}{2} I \omega^2 - \frac{mgl}{2} = 0$$

which gives

$$\omega = \sqrt{\frac{3g}{l}}$$

I now solve the problem by a direct application of torque equation. When the scale makes an angle θ from the horizontal (see figure 2), the torque on it is given as

$$\tau = \frac{mgl}{2} \cos \theta$$

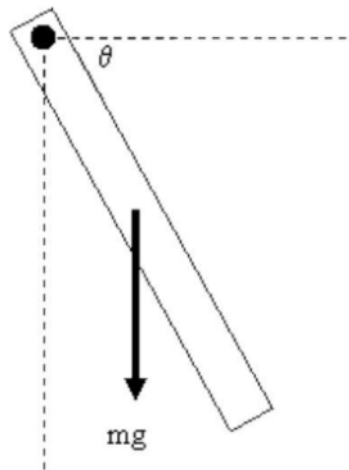


Figure 2

The angular momentum-torque equation then gives

$$I \frac{d\omega}{dt} = \frac{mgl}{2} \cos \theta$$

Substituting $\omega = \frac{d\theta}{dt}$ and the value of I from above this leads to

$$\frac{d^2\theta}{dt^2} = \frac{3g}{2l} \cos \theta$$

This equation cannot be integrated with respect to time directly. Recall from the proof of work-energy theorem that in such situations we change transform the equation to write it in terms of the displacement variable, which is the angle in this case. So we write

$$\frac{d^2\theta}{dt^2} = \frac{d}{dt} \left(\frac{d\theta}{dt} \right) = \frac{d\theta}{dt} \frac{d}{d\theta} \left(\frac{d\theta}{dt} \right) = \frac{1}{2} \frac{d}{d\theta} (\dot{\theta}^2)$$

to write the equation above as

$$\frac{1}{2} \frac{d\dot{\theta}^2}{d\theta} = \frac{3g}{2l} \cos \theta$$

Integrating this equation then gives

$$\dot{\theta}^2 = \frac{3g}{l} \sin \theta$$

For $\theta = \pi/2$ this gives the same answer as obtained earlier. If you have noticed, what we have done here is actually used the work-energy theorem

You may ask at this point: wouldn't the correct way of solving this problem be to equate the kinetic energy of the CM to the change in the potential energy. This would lead to

$$\frac{1}{2} m v_{CM}^2 = \frac{mgl}{2}$$

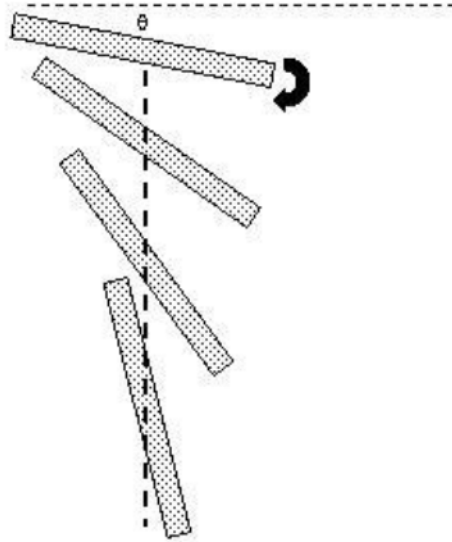
$$v_{CM} = \sqrt{gl} \quad \text{and} \quad \omega = \frac{v_{CM}}{l/2} = 2\sqrt{\frac{g}{l}}$$

The reason why this answer is incorrect is the following. Recall from our previous lecture that the most general motion of a rigid body is a translation plus a rotation. So while it is true that the CM is moving, the scale is also rotating at the same time. We represent the combination of the two motions as a translation of the CM and a rotation about an axis passing through the CM. Why we split the motion of the scale as a combination of the translation of its CM and a rotation about the CM - and not that of any other point in the body - will be discussed in detail below. For now it is sufficient to say that by doing so the kinetic energy can be written conveniently as (KE of the CM plus KE about the CM). So the true K.E of the scale is

$$\frac{1}{2}mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2$$

where $I_{CM} = \frac{ml^2}{12}$ is the moment of inertia about the CM. Using the relationship $v_{CM} = \frac{l}{2}\omega$ this gives the same kinetic energy as that used above in applying the energy conservation method. This correct approach then gives the same answer as obtained above.

An interesting problem related to the one solved above is as follows. Sometimes if a book you are holding slips out your hand, it usually falls with its upper face down (see figure 3). You can try this at home and see for yourself. In fact there is an interesting book which has a title based on this observation. It is entitled "Why toast lands jelly side down" and is authored by Robert Ehrlich (Universities press, Hyderabad 1999). Let us try to understand this observation.



A book falling down after slipping out of one's hand. Notice that the centre of the book is shown to be coming down in a straight line while the book rotates at a constant angular speed..

Figure 3

When the book falls its angular acceleration α immediately after it slips off the hand is calculated approximately as given below

$$\begin{aligned} I \frac{d\omega}{dt} &= \frac{mgl}{2} \Rightarrow \frac{ml^2}{3} \frac{d\omega}{dt} = \frac{mgl}{2} \\ \Rightarrow \alpha &= \frac{d\omega}{dt} = \left(\frac{3g}{2l} \right) \end{aligned}$$

Here m is the mass of the book and l its length. I call it an approximate expression because in our calculation we have assumed the book to be in horizontal position. It will slip off when $\tan \theta = \mu$ or for small angles $\theta \sim \mu$, where μ is the coefficient of friction between the book and the hand. Starting with zero initial angular speed, let the angular speed of the book when it slips out of the hand be ω . Then

$$\omega = \sqrt{2\alpha\theta} = \sqrt{\frac{3\mu g}{l}}$$

Taking $\mu = 0.5$, $g = 10\text{ms}^{-2}$ and $l = 20\text{cm} = 0.2\text{m}$, we get

$$\omega = 8.7 \text{ rad s}^{-1}$$

After the book has come out of the hands, there is no external torque on it about its CM so it falls rotating with a constant angular speed of about 8.7rad s^{-1} . Keep in mind that the sense and amount of rotation of a rigid body is the same irrespective of the point about which its rotation is considered. So although before slipping out of the hand, I did the calculation for its angular speed taking its edge on the hand as the axis, after it comes out of the hand, I consider its motion as the translation of its CM and rotation about its CM. Let us take a typical height of about 1m from which the book falls. Then the time it takes to reach the ground is

$$t = \sqrt{\frac{2h}{g}} = 0.45\text{s}$$

Thus the angle through which the book rotates by the time it reaches the ground is

$$\theta = \omega t = 8.7 \times 0.45 = 3.9\text{rad} \approx 225^\circ$$

If we add to this angle the initial rotation of $\theta = \mu = 0.5$, the angle increases to about 250° . The angle of rotation of course varies in a range but it is around 180° . You see that the book has just the right angular speed and the time of fall for it to turn by around 180° . That is precisely what we observe.

Rotation of a rigid body combined with translation of the axis parallel to itself:

Let us now introduce translation of the rotational axis parallel to itself - it may even accelerate - and ask what kind of motion is going to follow. So for example there may be a rod on a horizontal table and is hit by an impulse one end, and we may be interested in its subsequent motion. In general it could be a rigid body of general shape on which we apply a force. We split the motion into a translation of the CM of the body and rotation about an axis passing through the CM. By doing so the equation of motion for the translational motion of the CM is very easy. It is

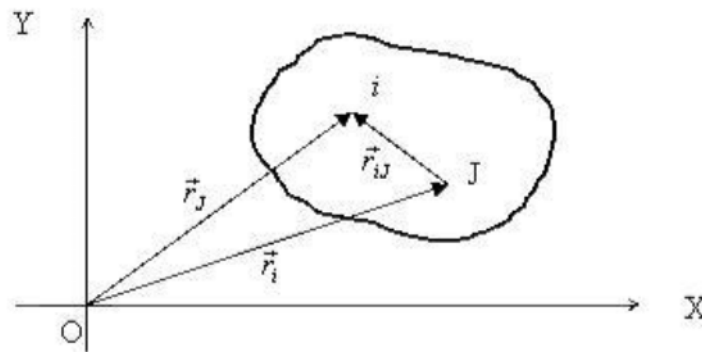
$$\frac{d\vec{p}_{CM}}{dt} = M\vec{a}_{CM} = \vec{F}_{applied}$$

Here \vec{p}_{CM} is the total momentum of the body; M is its mass; \vec{a}_{CM} the acceleration of the CM and $\vec{F}_{applied}$ the total applied force. With this equation we know how the CM of the body translates. Next we wish to find the rotation of the body with respect to an axis passing through the CM (recall that the most general motion of a rigid body is translation of a point and rotation about that point). But the question is: can we apply

$$\frac{dL_{CM}}{dt} = \tau_{applied,CM}$$

where L_{CM} is the angular momentum about the CM and $\tau_{applied,CM}$ is the applied torque about the axis of rotation passing through the CM. I raise this question because in general the CM will also be accelerating and therefore with respect to the CM, there will be a fictitious force that may

also give rise to a fictitious torque which is in addition to the applied torque $\tau_{applied,CM}$. However, it is easy to see that such a fictitious torque about the CM will always be zero. This is because the fictitious force effectively acts at the CM itself. Because of this reason, there is one more point about which the torque due to the fictitious force vanishes: this is the point that accelerates towards the CM. Thus the equation above can be applied safely about these two points. There is also a third point about which the above equation is valid. This is the point that does not accelerate at all. Let me now prove these statements.



A rigid body that is both translating and rotating. Two points in the body are shown by J and i.

Figure 4

Shown in figure 4 is a rigid body performing a general motion, i.e. it is both translating as well as rotating. For convenience we have shown the body in two dimensions. Two points J and i of the body are also shown. These points are also moving with the body. We now calculate the rate of change of the angular momentum about point J . This is done below.

$$\begin{aligned}\vec{L}_J &= \sum m_i \vec{r}_{iJ} \times \vec{v}_{iJ} \\ \frac{d\vec{L}_J}{dt} &= \sum m_i \vec{r}_{iJ} \times \frac{d\vec{v}_{iJ}}{dt} \\ &= \sum m_i \vec{r}_{iJ} \times \vec{a}_{iJ}\end{aligned}$$

where all the terms have their standard meaning and the subscript (iJ) denotes the quantity being calculated for point i with respect to point J . Denoting the velocity and acceleration of point i about the origin O as \vec{v}_i and \vec{a}_i , respectively, and that of J as \vec{v}_J and \vec{a}_J , we have

$$\vec{a}_{iJ} = \frac{d}{dt}(\vec{v}_i - \vec{v}_J) = \vec{a}_i - \vec{a}_J \quad \text{and} \quad \frac{d\vec{L}_J}{dt} = \sum_i m_i (\vec{r}_{iJ} \times \vec{a}_i) - \left(\sum_i m_i \vec{r}_{iJ} \right) \times \vec{a}_J$$

With $m_i \vec{a}_i = \vec{f}_{i, \text{applied}}$ and $\sum m_i \vec{r}_{iJ} = M \vec{R}_{CM,J}$, where $\vec{R}_{CM,J}$ is the position vector of the CM with respect to J , we get

$$\begin{aligned}\frac{d\vec{L}_J}{dt} &= \sum_i (\vec{r}_{iJ} \times \vec{f}_{i, \text{applied}}) - M \vec{R}_{CM,J} \times \vec{a}_J \\ &= \vec{\tau}_{\text{applied}} - M \vec{R}_{CM,J} \times \vec{a}_J\end{aligned}$$

If we want the rate of change of the angular momentum to depend only on the applied torque calculated about J , we should have

$$M \vec{R}_{CM,J} \times \vec{a}_J = 0$$

That will happen under the following three conditions:

- (I) $\vec{a}_J = 0$, i.e. J is moving uniformly,
- (II) $\vec{a}_J \parallel \vec{R}_{CM,J}$, i.e. J is accelerating towards the CM,
- (III) $\vec{R}_{CM,J} = 0$, i.e. point J is the CM

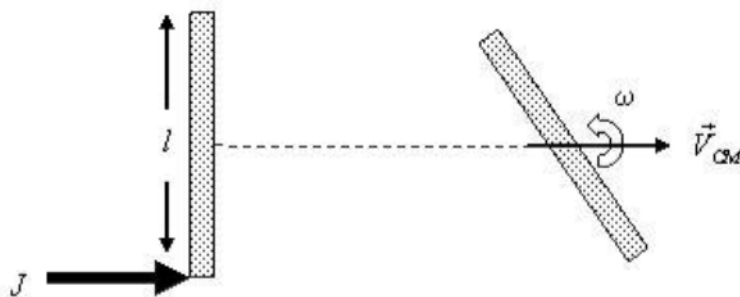
I have just shown you that irrespective of whether point J is accelerating, rotating or performing some general motion, the equation

$$\frac{d\vec{L}_J}{dt} = \vec{\tau}_{\text{applied}, J}$$

can be applied about J if it satisfies one of the three conditions obtained above. Notice that in under these conditions the right-hand side has only the externally applied torque. Thus if we choose one of these points to apply the angular momentum-torque equation, we do not have to

worry about any fictitious torques arising because we are sitting on an accelerating point. We have been applying the angular momentum-torque equation about points satisfying condition *I* above; it includes stationary points also. Of the other two points, it is always safer to apply the equation about the CM (condition *II*). This is because of the difficulty in ensuring that a point is accelerating towards the CM (condition *III*), although in some situations it may be easy. We will discuss one such case below. We now solve some simple examples to illustrate what we have learnt above.

Example 1: A uniform rod of mass m and length l is on a smooth horizontal table (friction = 0) and is hit at one of its ends so that an impulse J is imparted to it in its perpendicular direction (see figure 5). What is its subsequent motion?



A rod being hit at one of its ends giving it an impulse J (left) and its position and orientation of the rod some time after the impact (right).

Figure 5

As the rod is hit, its CM will start moving with a velocity

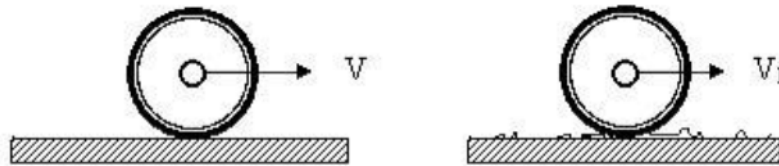
$$V_{\text{CM}} = \frac{J}{m}$$

At the same time the rod also starts rotating. Although the CM will be accelerating during the impact, we can apply the angular momentum-torque equation about it with only external torque in the equation. If the angular speed of the rod after the impact is ω , it is given by

$$\frac{l}{2} J = I_{\text{CM}} \omega \Rightarrow \omega = \frac{lJ}{2(ml^2/12)} = \left(\frac{6J}{ml} \right)$$

Note that in the sentence above, I have said 'angular speed of the rod' and not 'angular speed of the rod about the CM' because the sense and amount of rotation about any point in the body is the same, as was discussed in a previous lecture. The position and orientation of the rod some time after the impact is also shown in figure 5.

Example 2: A wheel of mass m and radius R is sliding on a smooth surface (No rolling) with speed V . It then hits a very rough surface so that it starts rolling (see figure 6). What is its rolling speed?



A wheel sliding on a smooth surface (left) hits a rough surface (right) and starts rolling.

Figure 6

Let the rolling speed of the wheel be V_1 . As soon as the wheel hits the rough surface, it gets an impulse J at its point on the surface in the direction opposite to its velocity. This reduces its speed and also makes it rotate. It rolls if the speed V_1 of its CM is equal to ωR , where ω is the rolling speed it gains after hitting the rough surface. The change in the CM speed is given by

$$V - V_1 = \frac{J}{M}$$

Applying the angular-momentum torque equation about the CM, we get

$$mR^2\omega = JR \Rightarrow J = mR\omega$$

With the condition of rolling, $V_1 = \omega R$, the above two equations give

$$V_1 = \frac{V}{2}$$

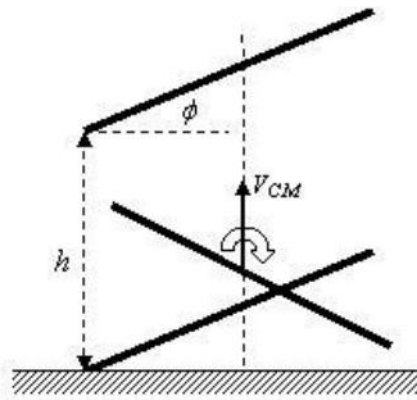
I would like you to repeat the same exercise for a disc.

The problem can also be solved by applying conservation of the point of impact on ground, because the impulse gives zero torque about that point. The initial angular momentum of the wheel with respect to that point is mVR . The final angular momentum is (angular momentum of the CM plus angular momentum about the CM). This comes out to be $(mV_1R + mR^2\omega)$. Equating this to mVR and using the rolling condition gives the same answer as above. A warning: keep in mind that the torque is being taken with respect to the point on ground and not the point

on the wheel that is touching the ground. Doing that will not be correct because at the time of impact the point on the wheel is accelerating in the direction opposite to \vec{v} .

I now solve a problem that involves, in addition to the equations above, energy conservation also.

Example 3: A rod of mass m and length l is held making an angle ϕ from the horizontal at a height h from the floor (see figure 7). When dropped from rest, what will be its linear and angular speed after it rebounds from the floor? Assume no energy is lost during the impact with the floor.



A rod dropped from a height rebounds from the floor and starts rotating. The CM moves only in vertical direction

Figure 7

When the rod hits the floor, it receives an impulse J from the ground in the vertically up direction. Although the rod is also being acted upon by its weight, we neglect its effect during impact (see discussion in the lecture on momentum). Since all the forces are in the vertical direction, the CM of the rod also moves only vertically. Before hitting the floor, the speed of the

CM is $\sqrt{2gh}$ and the angular speed of the rod is zero. Let the rebound speed of the CM be V and the angular speed of the rod after rebounding be ω . Then similar to the example above, these quantities are related as (keep in mind that we are dealing with vector quantities so their signs have to be properly accounted for)

$$mV + m\sqrt{2gh} = J$$

$$\frac{ml^2}{12} \omega = J \frac{l}{2} \cos \phi$$

These are two equations for three unknowns: V , ω and J . We therefore need one more equation. This is provided by energy conservation. We express the kinetic energy of the rod after it rebounds as the sum of the kinetic energy of its CM and the kinetic energy about its CM. Thus immediately after the impact, energy conservation gives

$$mgh = \frac{1}{2}mV^2 + \frac{1}{2}\frac{ml^2}{12}\omega^2$$

Now we have three equations that can be solved for the three unknowns. This is left for you to do.

Question you might now ask is if we could use the principle angular momentum and energy conservation directly to solve this problem in a manner similar to what we did at the end of the last example. We would like to apply the conservation of angular momentum about the point of impact on the ground because torque due to the impulse about this point vanishes. Although there is another external force - the weight of the rod - acting on the system, its effect during the impact can be ignored because very short duration of impact. Thus we can say that the angular momentum about the point of impact is conserved. This gives (left as an exercise for you)

$$m\sqrt{2gh}\frac{l}{2}\cos\phi = \frac{ml^2}{12}\omega - mV\frac{l}{2}\cos\phi$$

or

$$(m\sqrt{2gh} + mV)\frac{l}{2}\cos\phi = \frac{ml^2}{12}\omega$$

This is the same equation that is obtained by combining the first two equations above. Thus we obtain the same answer by this method also.

I end this lecture by giving you an exercise.

Exercise: A disc of mass m and radius R is made to roll on a rough surface by applying a force F at its centre. If it does not slip on the floor, i.e. it does pure rolling, find its acceleration by applying methods developed in this lecture.

Rotational dynamics IV: Angular velocity and angular momentum

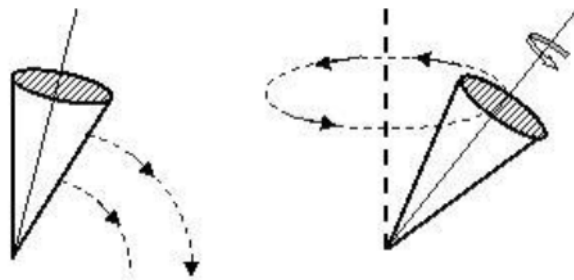
In the previous three lectures, we have dealt primarily with rotation about a fixed axis or an axis moving parallel to itself. What we saw in those lectures was that dynamics of a rigid body is

described by $\left(\frac{d\vec{L}}{dt}\right) = \vec{\tau}_{\text{external}}$ and in the absence of $\vec{\tau}_{\text{external}}$ the angular momentum \vec{L} is a conserved. In the case of fixed axis rotation, the relationship between the angular momentum and

the angular speed was quite straight forward in that $L = I\omega$ and all that was done in those problems was to change the magnitude of ω to change L . But the rotational motion is much more

interesting than that. For example \vec{L} is a vector so it could change direction because of applied torque with or without its magnitude being affected. How the changing direction of \vec{L} affects the orientation of a rigid body is one question we should answer if we wish to understand the motion of a rigid body. To start with, I want to point out to you that rotational motion is sometimes not what one would expect naively.

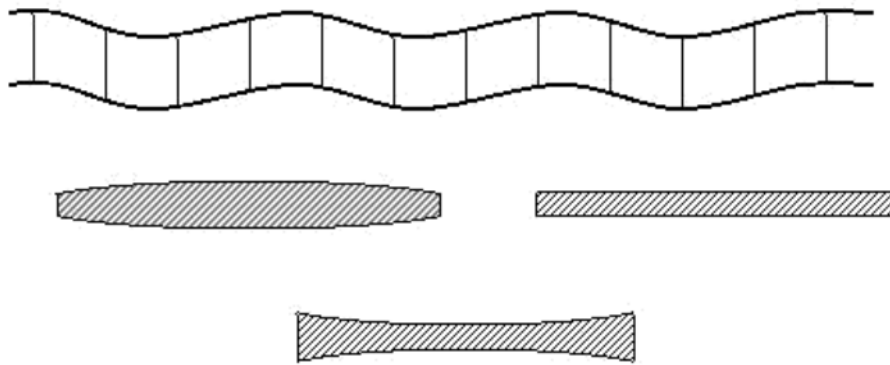
You must have played with a top. If it is not spinning and we try to make it stand on its pivot, it falls sideways. On the other hand, if it is given a spin and then put on its pivot point, it does not fall but starts to move about, what is called precession, a vertical axis passing through its pivot point. This is shown in figure 1. Obviously the precession of the top has something to do with its spin.



A top that is not spinning falls to the side (left) whereas a spinning top starts precessing (right).

Figure 1

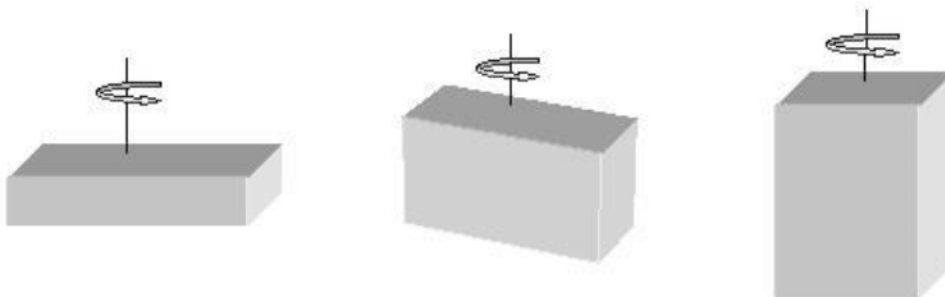
My second observation is from something that is seen in science museums. You can also make it easily in your local workshop. Take a track with many soft curves on it and let three different shape rollers roll on it. You may want to keep the track slightly tilted so that the rollers roll by themselves. Question is which of the rollers will be able to negotiate all the curves.



A curved track (top) as seen from above. Three rollers of different shapes are made to roll on it. Question is which one of these will negotiate all curves?

Figure 2

I make the third observation on a rectangular box of sweets (empty of course) or any similar box. Put a rubber-band around it so that its lid does not come off. Hold the box at a height with one of its faces perpendicular to the vertical, give it a spin and let it drop (see figure 3). Observe how its spin changes when it is falling down. You will find that in two out of three possible ways of holding the box, its spin will remain essentially unchanged whereas in one case it will start wobbling. On the other hand, if the box is dropped without giving it a spin, it comes down in the same orientation. What does the spin do to it? We wish to understand this.



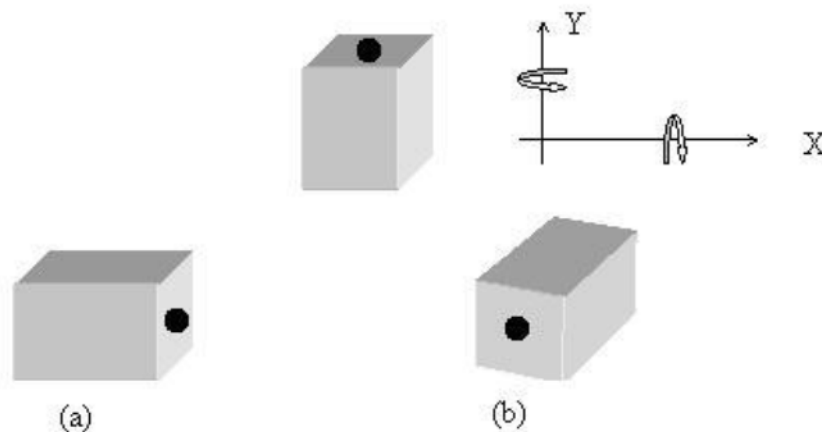
A box being given spin about different axes and being dropped from a height. In one case it wobbles while coming down.

Figure 3

In all three cases we see that when an object is given a spin its motion is very different compared to when it is not spinning. This happens because the angular momentum of the object due to its spin changes direction during the motion and the orientation of the body changes accordingly. So we now really have to get into the vector nature of angular momentum and relate it to the

parameters - the angle and the angular speed / velocity - of the body. I develop this structure of three-dimensional rigid-body dynamics step-by-step. The first question we address in this development is if the angle of rotation θ can be expressed as a vector $\vec{\theta}$? And if the answer is yes, what is its direction?

The answer to the question whether an angle of rotation can be treated as a vector is in the negative. This is because it fails to satisfy a fundamental property - that the addition of vectors is commutative - of vector addition. Thus if we make two rotations of angles θ_1 and θ_2 about two different axes, the end results will not be the same if the order of rotations is changed. This is depicted in figure 4 where I show a rectangular box that is to be rotated by 90° about the x and the y axes. The x and y axes are in the plane of the paper and pass through the centre of the box; the z -axis is coming out of the paper. The results are different if (a) I do the rotation about the x -axis first and then follow it with a rotation about the y -axis, and (b) I do the rotation about the y -axis first and then follow it with a rotation about the x -axis. Thus θ_1 and θ_2 cannot be treated as vectors because $\vec{\theta}_1 + \vec{\theta}_2 \neq \vec{\theta}_2 + \vec{\theta}_1$.

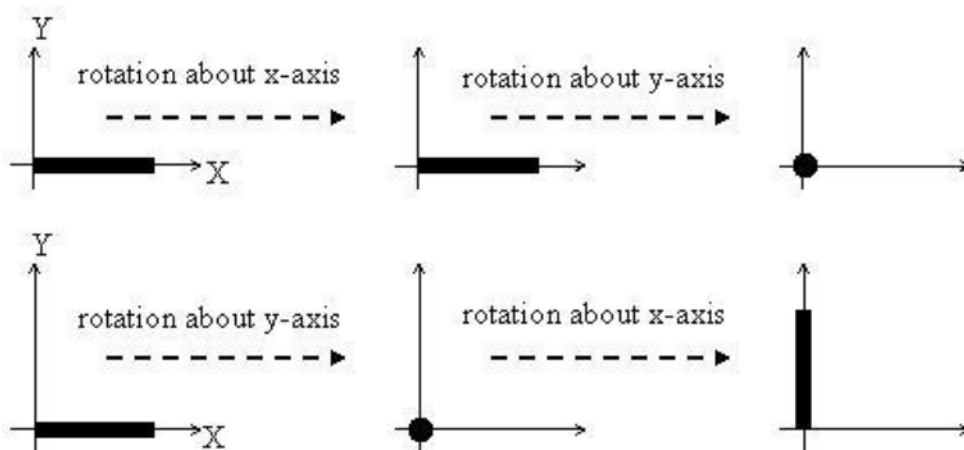


Results of two rotations – one about the x -axis by 90° counterclockwise and the second about the y -axis by 90° counterclockwise – performed on a box: (a) x -axis rotation that is followed by y -axis rotation, (b) y -axis rotation that is followed by an x -axis rotation.

Figure 4

Mathematically let us take a rod of length l lying along the x -axis with one of its ends at the origin so that the (xyz) coordinates of its other end are $(l, 0, 0)$. Keeping its end at the origin fixed, the rod is rotated about the x and the y axes in the same manner as the box in figure 4. If rotated about the x -axis first the end still has coordinates $(l, 0, 0)$. Now the rotation about the y -axis makes the rod align with the z -axis with the new coordinates of its end being $(0, 0, -l)$. Let us perform the rotations in the other order now. The first rotation is performed about the y -axis and makes the rod align with the z -axis with the new coordinates of its end being rod $(0, 0, -l)$

.Now the rotation about the x-axis makes the rod align with the y-axis and the final coordinates of its end are $(0, l, 0)$. Thus we see that two rotations have absolutely different effect on the orientation of a body depending on their order. This is demonstrated in figure 5. The conclusion therefore is that *rotations in general cannot be treated as vectors*.



Different order of rotation about x and y axes leaves the rod being rotated with two different final orientations. In the upper set of rotations, first rotation of 90° is performed about the x-axis and is followed by a rotation of 90° about the y-axis. In the lower set, the order is interchanged.

Figure 5

Although rotations by a finite angle are no vector quantities, rotations by infinitesimal angles

$\Delta\theta$ are. This also makes the derivative $\omega = \frac{\Delta\theta}{\Delta t}$ a vector quantity. We therefore call this quantity angular velocity rather than angular speed. Let me first show you through a simple example that infinitesimal rotations do satisfy the commutative property of vector addition and then go on to assign a direction to such rotations.

Let me again take a rod lying along the x-axis with one end fixed at the origin and the other at $(l, 0, 0)$. However, this time I consider infinitesimal rotations about the y and the z axes. I do so because I want both the rotations to cause change in the orientation of the rod; first rotation about the x-axis does not do that. Before I present the calculations, I would like you to recall from the first lecture how different components of a vector change when the frame is rotated. I would be making use of those relationships now with one change: rotating a vector by an angle $\Delta\theta$ about an axis is same as viewing it from a frame rotated by the angle $-\Delta\theta$ about the same axis. I perform a rotation of the rod about the y-axis by an angle θ_y and that about the z-axis by angle θ_z . Let me first consider the case of rotation about the y-axis that is followed by a rotation about the z-axis. Rotation of the rod about the y-axis gives the new coordinates of its free end as

$$\begin{aligned}
x' &= -0 \times \sin(-\Delta\theta_y) + l \cos(-\Delta\theta_y) \approx l \\
y' &= 0 \\
z' &= 0 \times \cos(-\Delta\theta_y) + l \sin(-\Delta\theta_y) \approx -l\Delta\theta_y
\end{aligned}$$

Now rotate the rod about z-axis to get coordinates of its free end as

$$\begin{aligned}
x'' &\approx l \cos(-\Delta\theta_z) + 0 \times \sin(-\Delta\theta_z) \approx l \\
y'' &= -l \sin(-\Delta\theta_z) + 0 \times \cos(-\Delta\theta_z) \approx l\Delta\theta_z \\
z'' &= -l\Delta\theta_y
\end{aligned}$$

Let us now do it the other way. Rotation about the z-axis gives

$$\begin{aligned}
x' &\approx l \cos(-\Delta\theta_z) + 0 \times \sin(-\Delta\theta_z) \approx l \\
y' &= -l \sin(-\Delta\theta_z) + 0 \times \cos(-\Delta\theta_z) \approx l\Delta\theta_z \\
z' &= 0
\end{aligned}$$

Now give a rotation about the y-axis to get

$$\begin{aligned}
x'' &\approx -0 \times \sin(-\Delta\theta_y) + l \cos(-\Delta\theta_y) \approx l \\
y'' &= l\Delta\theta_z \\
z'' &= 0 \times \cos(-\Delta\theta_y) + l \sin(-\Delta\theta_y) \approx -l\Delta\theta_y
\end{aligned}$$

When we compare the two boxed results above, we find that the coordinates of the end point of the rod come out to be the same. We conclude that two infinitesimal rotations will give the same final result irrespective of the order in which they are applied. Thus *infinitesimal rotations can be treated as vectors*. But what about the direction of rotation? To assign a direction, notice that the change in the position vector $\vec{r} = l\hat{x}$ of the end coordinate of the rod considered above can be written as

$$\begin{aligned}
\Delta\vec{r} &= (\Delta\theta_y \hat{j} + \Delta\theta_z \hat{k}) \times \vec{r} \\
&= (\Delta\theta_z \hat{k} + \Delta\theta_y \hat{j}) \times \vec{r} \\
&= l\Delta\theta_z \hat{i} - l\Delta\theta_y \hat{k}
\end{aligned}$$

where I have written the second line above to emphasize that the order in which infinitesimal rotations are performed does not affect the end result of these operations. The equations above suggest that an infinitesimal rotation about an axis be assigned a direction parallel to the axis following the right hand convention: If the thumb of the right hand points in the direction of the infinitesimal rotation, the movement of fingers gives the sense of rotation. With this definition, the change in the position vector of a point after it is rotated by an infinitesimal angle $\Delta\theta$ about an

axis in the direction of unit vector \hat{n} (sense of rotation given by right hand convention) is given as

$$\Delta \vec{r} = \Delta \theta \hat{n} \times \vec{r} = \Delta \vec{\theta} \times \vec{r}$$

It is obvious that the vector $\Delta \vec{\theta} = \Delta \theta \hat{n}$. The corresponding derivative with respect time is called the angular velocity, usually denoted by $\vec{\omega}$. Thus

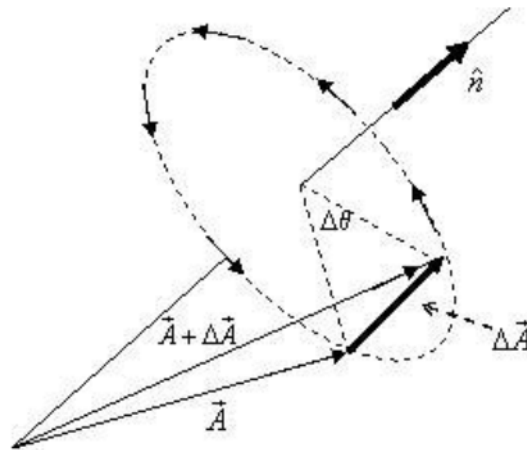
$$\vec{\omega} = \frac{d\theta}{dt} \hat{n}$$

I now point out that although the above equation is written for a position vector, there is nothing in its derivation that limits it to position vectors only. It is in fact true for any vector as can be easily proved by replacing the (xyz) coordinates by the corresponding components of the vector

in the derivation above. Thus if a vector \vec{A} is given an infinitesimal rotation $\Delta \vec{\theta} = \Delta \theta \hat{n}$, its will change by

$$\Delta \vec{A} = \Delta \vec{\theta} \times \vec{A}$$

This is shown pictorially in figure 6.



Change $\Delta \vec{A}$ in a vector \vec{A} when it is rotated by $\Delta \theta$ about an axis in the direction of \hat{n} .

Figure 6

Let us now see how much does a vector \vec{A} change when we apply two infinitesimal rotations $\Delta\vec{\theta}_1$ and $\Delta\vec{\theta}_2$ about two different axes. Let the vector be denoted by \vec{A}_1 after the first rotation and by \vec{A}_2 after the second one. Then we have

$$\begin{aligned}\vec{A}_1 &= \vec{A} + \Delta\vec{\theta}_1 \times \vec{A} \\ \vec{A}_2 &= \vec{A}_1 + \Delta\vec{\theta}_2 \times \vec{A}_1 \\ &= \vec{A} + \Delta\vec{\theta}_1 \times \vec{A} + \Delta\vec{\theta}_2 \times (\vec{A} + \Delta\vec{\theta}_1 \times \vec{A}) \\ &= \vec{A} + (\Delta\vec{\theta}_1 + \Delta\vec{\theta}_2) \times \vec{A} \text{ (up to first order in } \Delta\theta\text{)}\end{aligned}$$

thereby showing that for several infinitesimal rotations the final effect can indeed be expressed by adding the effect of each one of them.

Next we consider the rate of change of a vector rotating with an angular velocity $\vec{\omega}$. It is obtained as follows:

$$\begin{aligned}\Delta\vec{A} &= \Delta\vec{\theta} \times \vec{A} \\ \frac{\Delta\vec{A}}{\Delta t} &= \frac{\Delta\vec{\theta}}{\Delta t} \times \Delta\vec{A} \Rightarrow \frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A}\end{aligned}$$

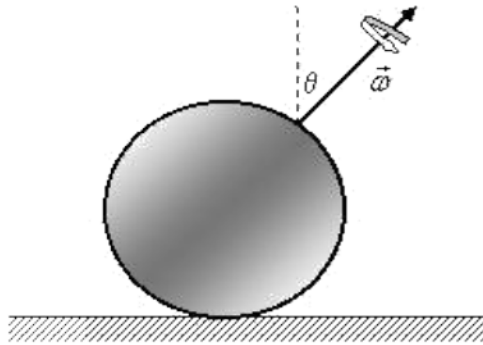
This is the rate of change of a vector \vec{A} only due to its rotation. If it changes additionally due to some other causes, that has to be added to the above change separately. If we take the vector \vec{A} to be the position vector \vec{r} , we get the formula

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$$

for linear velocity of a particle due to pure rotation of its position vector.

You may ask this point why is it that we want to take $\Delta\vec{\theta}$, $\vec{\omega}$ as vector quantities. The answer is that we in doing our calculations, we should know whether a quantity is a scalar or a vector or something else so that mathematical operations on it can be appropriately defined. For example, now that we know that $\vec{\omega}$ is a vector quantity, we can take its components and deal with them independently. Let me give you an example.

Example 1: A ball is given a spin at speed w and then put on a rough floor with $\vec{\omega}$ making an angle q with the vertical. When the ball eventually rolls, what would be its rolling speed (see figure 7)?



A spinning ball put on a rough floor with its angular velocity at an angle with the vertical.

Figure 7

In solving this problem, I make use of the vector nature of $\vec{\omega}$ and split it into its two components. It is the horizontal component $\omega \sin \theta$ that is responsible for making the ball roll. The vertical component $\omega \cos \theta$ does not contribute to rolling, as you well know. Further, this component eventually goes to zero due to friction. So the question is: if a sphere rotating with angular speed $\omega \sin \theta$ is kept on a rough floor with axis of rotation horizontal, what is its final rolling speed. I will let you figure that out. The point that is emphasized here is that knowing that $\vec{\omega}$ is a vector quantity helped us to solve the problem easily.

Now that we know $\vec{\omega}$ is a vector, the next question we ask is: how does $\vec{\omega}$ change when an external torque is applied on a body? So far we have learnt that external torques change angular momentum \vec{L} . So to know how $\vec{\omega}$ changes, we should know the relationship between \vec{L} and $\vec{\omega}$. We derive this relationship next.

Angular momentum of a rigid body rotating with angular velocity $\vec{\omega}$: We now derive the relationship between the angular momentum of a body rotating in space with one point fixed. That means the body is not translating and has only three degrees of freedom. By definition, the angular momentum

$$\vec{L} = \sum_i m_i \vec{r}_i \times \vec{v}_i$$

For a rigid body rotating with one point fixed, I have derived above that $\vec{v}_i = \vec{\omega} \times \vec{r}_i$. With

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i = (\omega_y z_i - \omega_z y_i) \hat{k} + (\omega_z x_i - \omega_x z_i) \hat{j} + (\omega_x y_i - \omega_y x_i) \hat{k}$$

we get

$$\begin{aligned}\vec{L} &= \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) \\ &= \sum_i m_i (x_i \hat{i} + y_i \hat{j} + z_i \hat{k}) \times \{(\omega_y z_i - \omega_z y_i) \hat{i} + (\omega_z x_i - \omega_x z_i) \hat{j} + (\omega_x y_i - \omega_y x_i) \hat{k}\}\end{aligned}$$

This gives the three components of the angular momentum to be

$$\begin{aligned}L_x &= \left(\sum_i m_i (y_i^2 + z_i^2) \right) \omega_x - \left(\sum_i m_i x_i y_i \right) \omega_y - \left(\sum_i m_i x_i z_i \right) \omega_z \\ &= I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z \\ L_y &= - \left(\sum_i m_i y_i x_i \right) \omega_x + \left(\sum_i m_i (z_i^2 + x_i^2) \right) \omega_y - \left(\sum_i m_i y_i z_i \right) \omega_z \\ &= I_{yx} \omega_x + I_{yy} \omega_y + I_{yz} \omega_z \\ L_z &= - \left(\sum_i m_i z_i x_i \right) \omega_x - \left(\sum_i m_i z_i y_i \right) \omega_y + \left(\sum_i m_i (x_i^2 + y_i^2) \right) \omega_z \\ &= I_{zx} \omega_x + I_{zy} \omega_y + I_{zz} \omega_z\end{aligned}$$

This is usually written in the matrix form

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

The (3 x 3) matrix in the equation above is known as the moment of inertia tensor. Its diagonal terms

$$I_{xx} = \sum_i m_i (y_i^2 + z_i^2) \quad I_{yy} = \sum_i m_i (z_i^2 + x_i^2) \quad I_{zz} = \sum_i m_i (x_i^2 + y_i^2)$$

are the moments of inertia about the x , y and the z -axis, respectively. The off-diagonal terms

$$I_{xy} = I_{yx} = - \sum_i m_i x_i y_i \quad I_{xz} = I_{zx} = - \sum_i m_i x_i z_i \quad I_{yz} = I_{zy} = - \sum_i m_i y_i z_i$$

are known as the products of inertia. The values of the moments and products of inertia depend on the set of axes chosen.

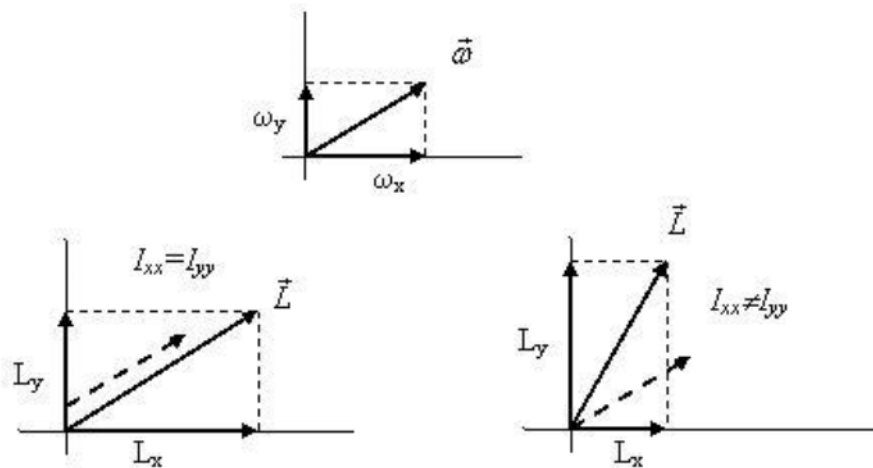
So you see that relationship between \vec{L} and $\vec{\omega}$ is quite involved. Luckily, for a rigid body, for each point one can find a set of axes about so that products of inertia about that point vanish. These are known as the principal axes. Thus for the principal set of axes at a point

$$I_{xy} = I_{yx} = I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0$$

These axes are attached with the body and rotate with it. However, the principal axes offer an advantage when dealing with the angular momentum of a rigid body. At a given time, if I calculate the components of the angular momentum by taking the rigid-body to be rotating in the principal axes frame at that instant, they turn out to be simply $L_x = I_{xx}\omega_x$, $L_y = I_{yy}\omega_y$ and $L_z = I_{zz}\omega_z$. Thus the angular momentum of the body is given as

$$\vec{L} = I_{xx}\omega_x\hat{i} + I_{yy}\omega_y\hat{j} + I_{zz}\omega_z\hat{k}$$

at any given instant. It is easily seen from the expression above that in general the angular momentum and the angular velocity are not parallel; they will be parallel only if $I_{xx} = I_{yy} = I_{zz}$, i.e. if all three moments of inertia about the principal axes are equal. This is shown in figure 8 in two dimensions.

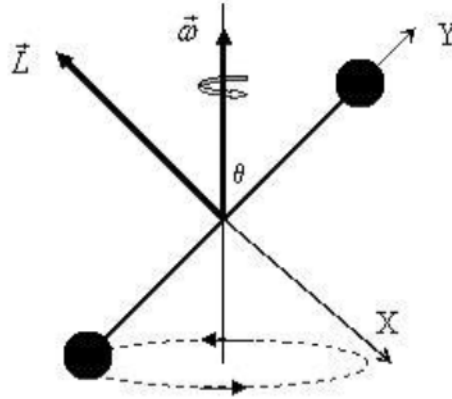


Direction of \vec{L} for a given $\vec{\omega}$ (upper figure). If $I_{xx} = I_{yy}$, \vec{L} and $\vec{\omega}$, shown by dashed arrow, are parallel (lower left). If $I_{xx} \neq I_{yy}$, \vec{L} and $\vec{\omega}$ are not parallel (lower right).

Figure 8

Let me now solve an example.

Example 2: A thin massless rod of length $2l$ has a point mass m at both its ends. It is rotating with angular speed ω about a vertical axis passing through its centre and at an angle θ from it, as shown in figure 9. Calculate its angular momentum.



A thin massless rod with point mass m at both its ends rotating about a vertical axis. The principal axes are shown as X and Y axis with Z perpendicular to the XY plane.

Figure 9

We will apply the formula for angular momentum derived above. It is easy to see that at the centre of the rod, the principal axes are: one axis parallel to the rod and two of them perpendicular to it. These are shown in the figure above. Notice that the principal axes rotate with the body. The moment of inertia with respect to the principal axes shown in figure 9 are

$$I_{xx} = 2ml^2 \quad I_{yy} = 0 \quad I_{zz} = 2ml^2$$

The components of the angular velocity along the principal axes are

$$\omega_x = -\omega \sin \theta \quad \omega_y = \omega \cos \theta \quad \omega_z = 0$$

Thus the angular momentum is given as

$$\begin{aligned} \vec{L} &= I_{xx} \omega_x \hat{i} + I_{yy} \omega_y \hat{j} + I_{zz} \omega_z \hat{k} \\ &= -2ml^2 \omega \sin \theta \hat{i} \end{aligned}$$

This is also shown in figure 9. It is clear from the figure that as the body rotates so does its angular momentum vector. Thus the angular momentum of the body changes with time although its magnitude remains unchanged.

I end this lecture by asking you to solve a similar problem.

Exercise: A rectangular thin sheet of sides a and b is rotating about one of its diagonals (see figure 10) with angular speed ω . The mass of the sheet is m . What is its angular momentum? Express it in terms of the principal axes unit vectors.

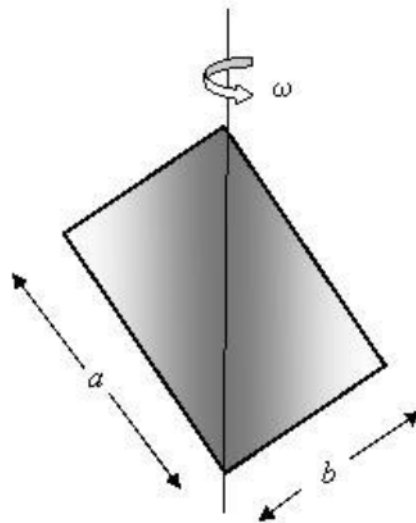


Figure 10

Rotational dynamics V: Kinetic energy, angular momentum and torque in 3-dimensions

You learnt in the previous lecture is that the angular velocity $\vec{\omega}$ is a vector quantity pointing in the direction of the axis of rotation. Any vector that is rotating about $\vec{\omega}$ also changes direction.

Thus the vector changes even if its magnitude is constant. If the vector is \vec{A} then its rate of change purely on the basis of rotation is

$$\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A}$$

Thus the velocity of a rotating particle at position \vec{r}_i from the origin is

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i$$

I also derived the general expression for the angular momentum, which is given as

$$\begin{aligned}\vec{L} = & (I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)\hat{i} \\ & + (I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z)\hat{j} \\ & + (I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z)\hat{k}\end{aligned}$$

Here I_{xx} , I_{yy} and I_{zz} are the moments of inertia about the x , y and the z axes, respectively. The off diagonal elements like I_{xy} are the products of inertia. A simplification in the expression above arises by employing the principal axes for which the products of inertia vanish. For convenience in writing, the principal axes are usually denoted by $(1,2,3)$ instead of (x,y,z) . Using this notation the angular momentum vector can be written in a simple form as

$$\vec{L} = I_1\omega_1\hat{i} + I_2\omega_2\hat{j} + I_3\omega_3\hat{k}$$

where ω_1 , ω_2 and ω_3 are the components of the angular velocity along the principal axes. I now derive the expression for kinetic energy for a rigid body rotating with one point fixed.

Kinetic energy of a rotating rigid body: I consider a rigid body rotating with angular velocity $\vec{\omega}$. Its kinetic energy T is calculated as follows

$$T = \frac{1}{2} \sum_i m_i \vec{v}_i \cdot \vec{v}_i$$

Substituting $\vec{v}_i = \vec{\omega} \times \vec{r}_i$ for the velocities above and making use of some identities of vector products we get

$$\begin{aligned}T &= \frac{1}{2} \sum_i m_i \vec{v}_i \cdot (\vec{\omega} \times \vec{r}_i) \\ &= \frac{1}{2} \sum_i m_i \vec{r}_i \cdot (\vec{v}_i \times \vec{\omega}) \\ &= \frac{1}{2} \sum_i m_i \vec{\omega} \cdot (\vec{r}_i \times \vec{v}_i) \\ &= \frac{1}{2} \vec{\omega} \cdot \vec{L}\end{aligned}$$

In the principal axes therefore

$$T = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2$$

This is the expression for the kinetic energy in terms of the principal moments of inertia and the components of angular velocity along the principal set of axes. Having obtained the general expressions for the angular momentum and kinetic energy of a rigid body, we now study the dynamics of a rigid body through the angular-momentum torque equation. Along the way I will explain the three observations that I had started my previous lecture with.

Dynamics of a rigid body: Dynamics of a rigid body is governed by the equation

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\text{applied}}$$

and it is this equation that governs everything about the rigid-body rotation. What makes the motion of a rigid-body interesting is that there is a fantastic interplay between the angular momentum, angular velocity of a rigid body with or without an applied torque. For example if the angular velocity and the angular momentum of a rigid body are not parallel, the \vec{L} vector would rotate about $\vec{\omega}$ and that would make \vec{L} change. However, if there is no torque applied on the body, angular momentum cannot change. Therefore to compensate the change in \vec{L} arising from its rotation, the angular velocity $\vec{\omega}$ itself must change. Changing $\vec{\omega}$ would make body rotate in a different way and this goes on. It is thus this interplay between \vec{L} and $\vec{\omega}$ that makes a rigid body move in seemingly counterintuitive ways.

As a body rotates, its angular momentum changes on two counts: first because in general \vec{L} and $\vec{\omega}$ are not parallel and therefore \vec{L} rotates about $\vec{\omega}$. With

$$\vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$$

and

$$\vec{L} = I_1 \omega_1 \hat{i} + I_2 \omega_2 \hat{j} + I_3 \omega_3 \hat{k}$$

the rate of change of \vec{L} only due to its rotation about $\vec{\omega}$ is given as

$$\begin{aligned} \frac{d\vec{L}}{dt} &= \vec{\omega} \times \vec{L} \\ &= (\omega_2 L_3 - \omega_3 L_2) \hat{i} + (\omega_3 L_1 - \omega_1 L_3) \hat{j} + (\omega_1 L_2 - \omega_2 L_1) \hat{k} \end{aligned}$$

UNIT – V

MECHANICAL VIBRATIONS

If the components ω_1, ω_2 and ω_3 were also changing, I would have to add an additional term on the right-hand side of the expression above to take care of that. This is the second reason for the change in angular momentum of the body. For the time being I focus on cases where the components of $\vec{\omega}$ along the principal axis remain unchanged. This in turn implies that the magnitude of the angular momentum remains constant during the rotational motion of the body. This happens when the applied torque is always perpendicular to the angular momentum. Substituting for L_1, L_2 and L_3 in the equation above, I get

$$\frac{d\vec{L}}{dt} = \omega_2 \omega_3 (I_3 - I_2) \hat{i} + \omega_3 \omega_1 (I_1 - I_3) \hat{j} + \omega_1 \omega_2 (I_2 - I_1) \hat{k}$$

So at any instant the components of $\frac{d\vec{L}}{dt}$ are

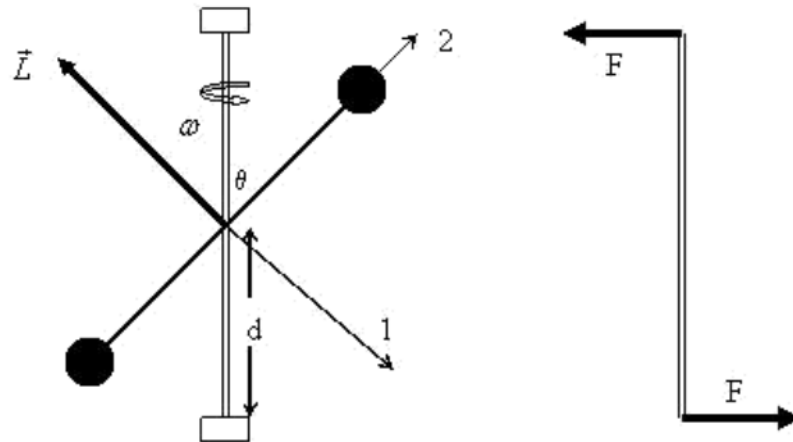
$$\left(\frac{d\vec{L}}{dt}\right)_1 = \omega_2 \omega_3 (I_3 - I_2) \quad \left(\frac{d\vec{L}}{dt}\right)_2 = \omega_3 \omega_1 (I_1 - I_3) \quad \text{and} \quad \left(\frac{d\vec{L}}{dt}\right)_3 = \omega_1 \omega_2 (I_2 - I_1)$$

For a geometric interpretation of these equations I urge you to go back to the previous lecture and see how we obtained the changes in the coordinates of the end of a rod rotating infinitesimally. This gives the components of the torque required to be

$$\begin{aligned} \tau_1 &= \omega_2 \omega_3 (I_3 - I_2) & \tau_2 &= \omega_3 \omega_1 (I_1 - I_3) \\ \tau_3 &= \omega_1 \omega_2 (I_2 - I_1) \end{aligned}$$

To apply these equations I start with calculation of torque for the example that we solved at the end of the previous lecture.

Example 1: A thin massless rod of length $2l$ has a point mass m at both its ends. It is rotating with angular speed ω about a vertical axis passing through its centre and at an angle θ from it, as shown in figure 1. If the axis of rotation is held at its two ends by ball bearings, calculate the force that the ball bearings apply on the axis. The ball bearings are placed symmetrically from the centre of the rod at a distance d each.



A thin massless rod with point mass m at both its ends rotating about a vertical axis (left). The axis of rotation is kept fixed in place by two ball bearings at a distance of d from the centre of the rod. The forces on the rod applied by bearings are also shown (right).

Figure 1

Recall from the previous lecture that I had taken the principal axes $(1,2,3)$ with $(1,2)$ as shown in figure 1 and axis 3 perpendicular to them. The moments of inertia about the principal axes are

$$I_1 = 2ml^2 \quad I_2 = 0 \quad \text{and} \quad I_3 = 2ml^2$$

The angular velocity and the angular momentum of the rod-mass system are

$$\begin{aligned} \vec{\omega} &= \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k} \\ &= -\omega \sin \theta \hat{i} + \omega \cos \theta \hat{j} \end{aligned}$$

and

$$\begin{aligned} \vec{L} &= I_1 \omega_1 \hat{i} + I_2 \omega_2 \hat{j} + I_3 \omega_3 \hat{k} \\ &= -2ml^2 \omega \sin \theta \hat{i} \end{aligned}$$

All the parameters - mass m , length l and angle θ - in the equation above are constant so the magnitude of the angular momentum is also a constant. As such we can apply the formulae given above to get the components of the torque to be applied as

$$\begin{aligned} \tau_1 &= \omega_2 \omega_3 (I_3 - I_2) = 0 & \tau_2 &= \omega_3 \omega_1 (I_1 - I_3) = 0 \\ \tau_3 &= \omega_1 \omega_2 (I_2 - I_1) = 2ml^2 \omega^2 \sin \theta \cos \theta \end{aligned}$$

Thus the torque needed to keep the rotating rod in its position is in the direction of principal axis 3 of the body. As was noted above, the torque is indeed perpendicular to \vec{L} . The torque is provided by the forces applied by the bearings. When the rod is in the plane of the paper, as shown in the figure, the force would be to the left at the upper end and to the right at the lower end of the rod (see figure 1). And their magnitudes will be equal since the CM of the rod has zero acceleration. Thus the forces provide a couple equal to τ_3 . Their magnitude is

$$F = \frac{2ml^2 \omega^2 \sin \theta \cos \theta}{2d} = \frac{ml^2 \omega^2 \sin \theta \cos \theta}{d}$$

There is another method of calculating $\frac{d\vec{L}}{dt}$ that we describe now. \vec{L} has one component $L_V = 2ml^2 \omega \sin^2 \theta$ in the direction of $\vec{\omega}$ and the other component $L_H = 2ml^2 \sin \theta \cos \theta$ perpendicular to $\vec{\omega}$ (see figure 2).

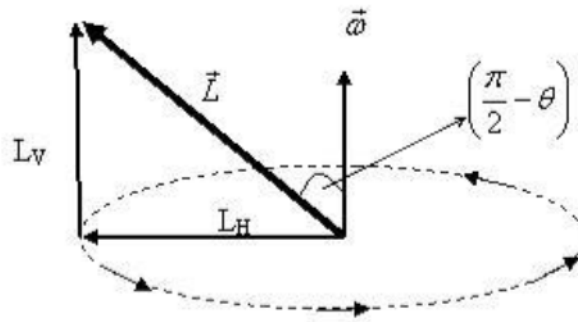


Figure 2

As the rod rotates L_V remains unchanged but L_H sweeps a circle with angular frequency ω . The rate of change of \vec{L} is therefore the same as that of L_H . The magnitude of the latter is ωL_H . Since at the position shown, the tip of L_H is moving out of the paper, the direction of the change in L_H is also the same. This is the direction of principal axis 3. It thus follows that

$$\begin{aligned} \left| \frac{d\vec{L}}{dt} \right| &= \omega L_H \\ &= 2ml^2 \omega^2 \sin \theta \cos \theta \end{aligned}$$

in the direction of principal axis 3. For completeness I also calculate the kinetic energy of the rod-mass system. It is

$$T = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_2^2$$

$$= ml^2 \omega^2 \sin^2 \theta$$

I now give you a couple of exercises similar to the problem above.

Exercise 1: In the problem above, if the axis of rotation passes through a different point than the centre of the rod (see figure 3), what will be the forces applied by the bearings with everything else remaining the same? (*Hint: the CM is now moving in a circle*)

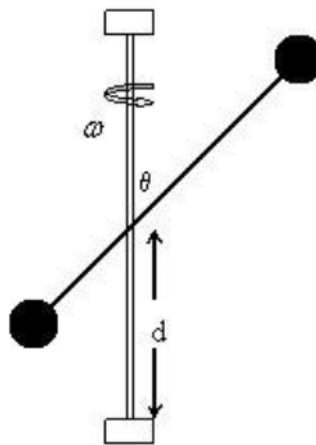
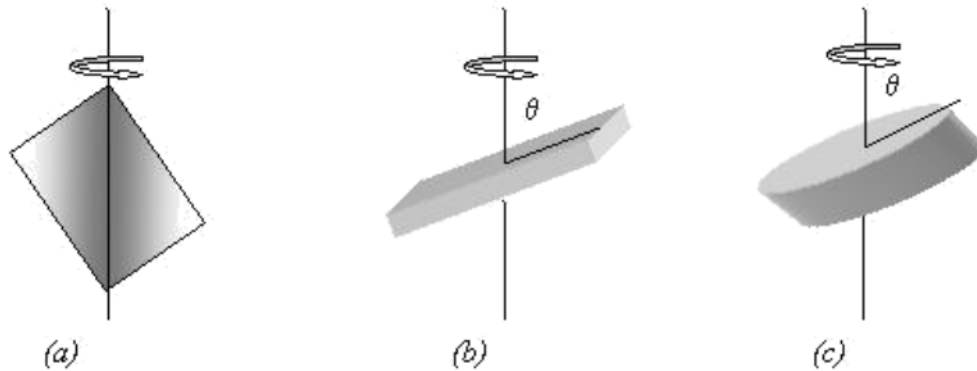


Figure 3

Exercise 2: For the rotating objects shown below in figure 4, calculate the rate of change of their angular momentum by the two methods employed in the example above.

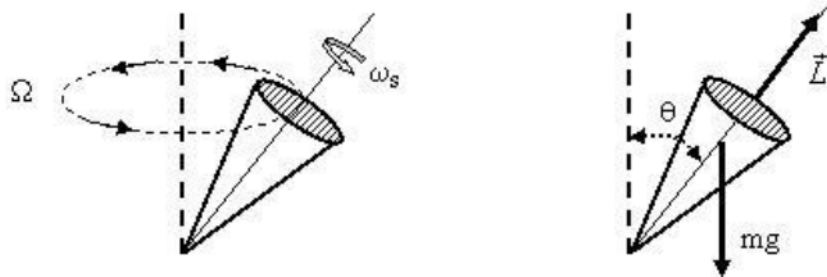


Find the rate of change of angular momentum of (a) a rectangular sheet rotating about its diagonal, (b) a rectangular sheet rotating about an axis passing through its centre, and (c) a thin disc rotating about an axis passing through its centre.

Figure 4

If you have followed the example above, and have also done the exercises suggested, then you will be in a position to understand the explanation of two of the three observations I started my previous lecture with. The two observations were the precession of a spinning top and only one roller of the three shown being able to go over a curved track entirely.

Example 2: Let me take the case of the precession of a spinning top. In this case we observe that when a spinning top is put on a floor and its lower point is held at one point, it starts precessing about the vertical axis (see figure 5)



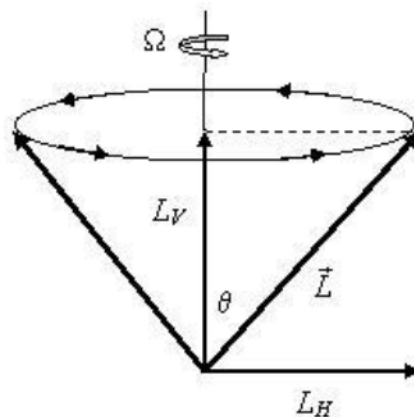
A precessing top (left) and its angular momentum and force mg acting on it (right).

Figure 5

I take the mass of the top to be m , its moment of inertia about the spinning axis I , distance of its CM from the pivot point l and its spinning rate to be ω_s . The top's axis is making an angle θ from the vertical. Let us take the rate of precession, i.e. the angular speed at which the top starts to

rotate about the vertical to be Ω . It is observed that Ω is usually much smaller than ω_s . So in calculating angular momentum we are going to take it as arising from the spin only and neglect any contribution of Ω to it. The angular momentum is then along the spin axis of the top and its magnitude is $L = I\omega_s$, where I is top's moment of inertia about its axis. Further, there is torque acting on the top due to its weight. The magnitude of the torque is $mg\sin\theta$ and it is perpendicular to the plane formed by the vertical and the spin axis (the direction of \vec{L}). At the position shown in figure 5, the torque is going into the plane of the paper. The problem then reduces to the following. A rigid body has an angular momentum \vec{L} and is being acted upon by a torque of magnitude $mg\sin\theta$ perpendicular to \vec{L} . What will happen to the body?

Since the angular momentum is being acted upon by a torque perpendicular to it, it changes continuously with time with its magnitude remaining unaffected. Thus it moves on the surface of a cone as shown in figure 6.



Angular momentum vector moving on the surface of a cone at a constant rate. Its vertical and horizontal components are L_v and L_H , respectively.

Figure 6

Let me now calculate the frequency of rotation of vector \vec{L} . For this I again look at the vertical L_v and horizontal L_H components of the angular momentum, as shown in figure 6. The vertical component remains unchanged and the horizontal component changes at the rate ΩL_H as the \vec{L}

$$\left(\frac{dL}{dt} \right) = \Omega L \sin \theta$$

vector rotates. This gives , which should be equal to the torque. Substituting $L = I\omega_s$, I thus get

$$\Omega I \omega_s \sin \theta = mgl \sin \theta$$

$$\Omega = \frac{mgl}{I \omega_s}$$

This is the rate at which the \vec{L} vector rotates. Since \vec{L} is attached to the top, the top also rotates at

the same rate. $\Omega = \frac{mgl}{I \omega_s}$ is then the rate of precession of the cone.

As the top precesses, its CM moves in a circle. You may now wonder where does the centripetal force for this come from? This is provided by the horizontal reaction or the frictional force at the pivot point. Second question you may raise is why is it that the component L_H starts moving in a horizontal circle due to the torque while the vertical component does not move in a vertical circle. In the actual motion, it does. So in addition to the precession, the axis of the top also oscillates up and down with very small amplitude. If you are careful in your observations, you will see this motion. This is known as the nutation of the top. In our present treatment, we have ignored this motion and solved the problem only to get the precession rate.

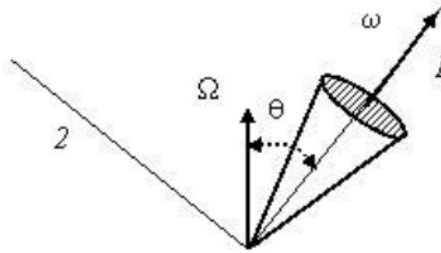
I now wish to explore if to get this answer, I could equivalently have used the equations

$$\tau_1 = \omega_2 \omega_3 (I_3 - I_2)$$

$$\tau_2 = \omega_3 \omega_1 (I_1 - I_3)$$

$$\tau_3 = \omega_1 \omega_2 (I_2 - I_1)$$

To do this, let me first identify the principal axes of the cone at the pivot point and label them. The principal axes are the spin axis and two other axes perpendicular to it. These are shown and labeled (1,2,3) in figure 7; in this position axes 1 and 2 are in the plane of the paper and axis 3 is coming out of it.



A spinning top precessing about the vertical. Its principal axes 1 and 2 are shown.

Figure 7

The moments of inertia about the principal axes are $I_1 = I, I_2 = I_3 = I_{\perp}$. The components of $\vec{\omega}$ at the instant (I take it to be time $t = 0$) shown in figure 7 are

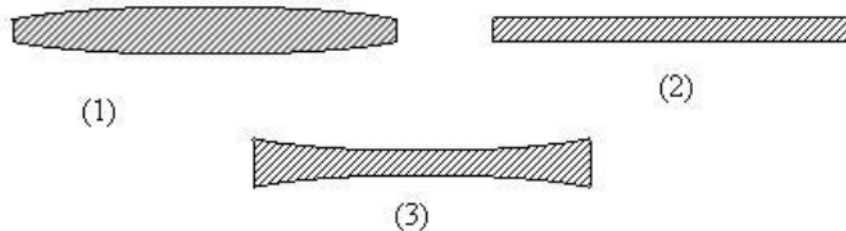
$$\begin{aligned}\omega_1 &= \omega_s + \Omega \cos \theta \approx \omega_s, & \omega_2 &= \Omega \sin \theta \\ \omega_3 &= 0\end{aligned}$$

Substituting the values of moments of inertia and the angular velocity components in the equations for the components of the torque gives

$$\begin{aligned}\tau_1 &= 0 & \tau_2 &= 0 \\ \tau_3 &= \Omega \omega_s \sin \theta (I_{\perp} - I)\end{aligned}$$

This is not the same answer as obtained earlier. Where have we gone wrong? Is the previous answer correct or is this answer correct? We will see later that in applying the equations above, we have not taken into account the fact that due to the spin of the top, its principal axes also spin about axis I and that makes the components of $\vec{\omega}$ along them time-dependent. For now I move on to explain the observation about only one of the rollers being able to go over all the curves of a track.

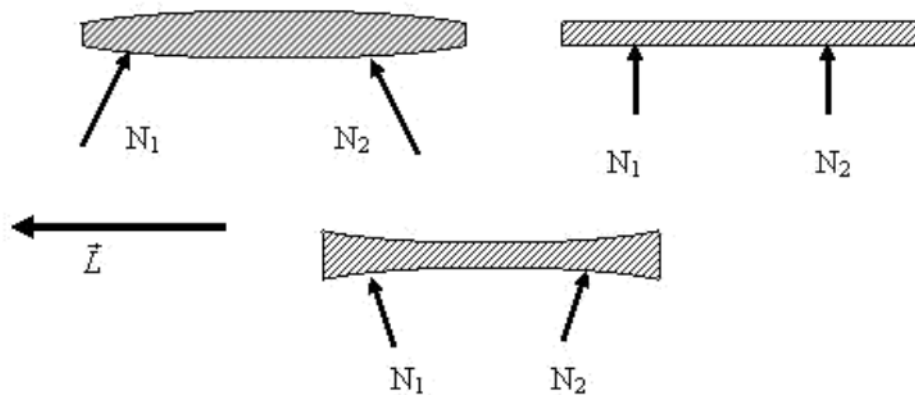
Example 3: If you have performed the experiment, you would have seen that only roller 1 (see figure 8) that is tapering down as we move away from its centre is able to go over all the curves. Let me now explain that.



Three rollers of different shapes that are made to roll on a curved track. Only roller 1 goes over the entire track; the other two fall off it.

Figure 8

As a roller goes over a curve, its centre of mass moves requires a centripetal force to do so. At the same time, the angular momentum of the roller also changes direction and that requires a torque. Both the centripetal force and the torque are provided by the normal reaction of the track on the rollers. These reaction forces on the three rollers are shown in figure 9.



Normal reaction on three rollers of figure 8. The rollers are moving into the page. Their angular momentum is pointing to the left.

Figure 9

In analyzing the motion of these rollers, I am taking them to be moving into the paper. Thus the direction of their angular momentum is to the left, as shown in the figure. Now if these rollers have to make a turn, the normal reactions should provide the required centripetal force in the horizontal direction. This rules out the plain cylindrical roller (roller 2) from making any turn because both normal reactions on it are in the vertical direction. This leaves the other two cylinders for further consideration. For those rollers, the torque of the normal reaction forces about the CM should change their angular momentum vector in the appropriate direction. Let us look at roller 1 first.

Roller 1: For a left turn, $N_1 < N_2$ for centripetal force. Therefore the torque generated by them is in the direction coming out of the page. As the roller makes a left turn, the associated change in its angular momentum also is in the direction coming out of the page, consistent with the torque generate. For a right turn by this roller, the centripetal force is to the right so $N_1 > N_2$. This generates a torque about the CM that goes into the page. For the right turn, the change in the angular momentum is also into the page, consistent with the torque generated. Thus for roller 1, the centripetal force and the torque generated are consistent with the centripetal force and the change in its angular momentum. Let us now see what happens to roller 3.

Roller 3: If roller 3 turns left, the centripetal force will be provided correctly if $N_1 > N_2$. This however gives a torque about the CM that is going into the page. On the other hand, during left turn the change in the angular momentum comes out of the page. Thus the torque and the change in angular momentum are in opposite directions. Exactly the same situation arises for a right turn. Because of this inconsistency, the roller fails to turn at any of the curves. This example teaches us about the centre of mass motion combined with angular momentum changes about the CM. We now move on to discuss the general form of the equation relating the torque and the angular momentum.

The general equation governing rotation of a rigid body:

Having dealt with situations where components of $\vec{\omega}$ are constant, we now ask what happens when $\vec{\omega}$ is also changed. For this let me look at the expression for the angular momentum in the principal axis frame again. It is

$$\vec{L} = I_1 \omega_1 \hat{i} + I_2 \omega_2 \hat{j} + I_3 \omega_3 \hat{k}$$

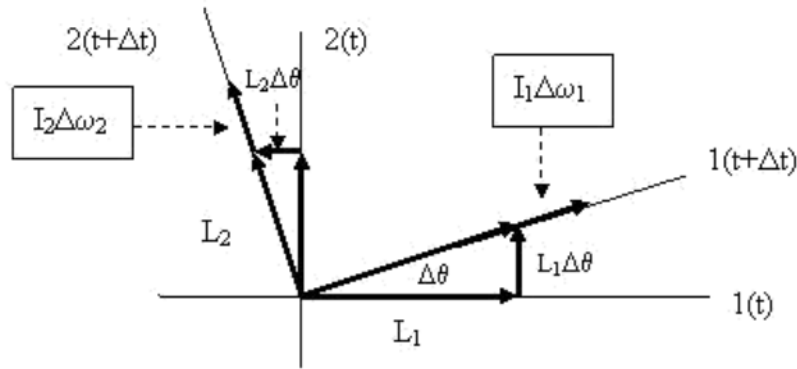
I now give a slightly different derivation for the rate of change of \vec{L} . In doing this derivation I keep in mind that as a rigid body rotates, the unit vectors along its principal axes also rotate and their rate of change is (see previous lecture)

$$\frac{d\hat{i}}{dt} = \vec{\omega} \times \hat{i} \quad \frac{d\hat{j}}{dt} = \vec{\omega} \times \hat{j} \quad \frac{d\hat{k}}{dt} = \vec{\omega} \times \hat{k}$$

Now I differentiate \vec{L} to get

$$\begin{aligned} \frac{d\vec{L}}{dt} &= \left(I_1 \frac{d\omega_1}{dt} \hat{i} + I_2 \frac{d\omega_2}{dt} \hat{j} + I_3 \frac{d\omega_3}{dt} \hat{k} \right) + \left(I_1 \omega_1 \frac{d\hat{i}}{dt} + I_2 \omega_2 \frac{d\hat{j}}{dt} + I_3 \omega_3 \frac{d\hat{k}}{dt} \right) \\ &= \left(I_1 \dot{\omega}_1 \hat{i} + I_2 \dot{\omega}_2 \hat{j} + I_3 \dot{\omega}_3 \hat{k} \right) + \vec{\omega} \times \left(I_1 \omega_1 \hat{i} + I_2 \omega_2 \hat{j} + I_3 \omega_3 \hat{k} \right) \\ &= \left(I_1 \dot{\omega}_1 \hat{i} + I_2 \dot{\omega}_2 \hat{j} + I_3 \dot{\omega}_3 \hat{k} \right) + \vec{\omega} \times \vec{L} \end{aligned}$$

Here the first term is due to the change in the components of $\vec{\omega}$ along the principal axis and the second term is the change in \vec{L} due to its rotation. Notice that we recover the formula derived earlier if the components of $\vec{\omega}$ do not change with time, i.e. $\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$. Let me repeat the interpretation of the equation: at any instant we take the body rotating in the principal axes frame at that time, i.e. the frame is frozen at its position at that time and the body is taken to be rotating in it. To see this geometrically, let me take a two-dimensional case. Shown in figure 10 are the principal axes 1 and 2 of a rigid body at times t and $(t + \Delta t)$. In time interval Δt the body and the frame attached to it rotate by an angle $\Delta \vec{\theta} = \Delta \theta \hat{k}$, and ω_1 and ω_2 change to $\omega_1 + \Delta \omega_1$ and $\omega_2 + \Delta \omega_2$. With these changes let me calculate changes in the components L_1 and L_2 in the frame frozen at time t .



The principal axes of a rotating body at time t and $(t+\Delta t)$. In time interval Δt the body and the attached frame have rotated by an angle $\Delta\theta$ about the z -axis and the components of angular velocity have changed by $\Delta\omega_1$ and $\Delta\omega_2$.

Figure 10

Looking at the figure, where I have shown all the changes that have taken place during the time interval Δt , we get in the frame at time t

$$\begin{aligned}\Delta L_1 &= I_1 \Delta \omega_1 \cos \Delta \theta - L_2 \Delta \theta \\ &= I_1 \Delta \omega_1 - L_2 \Delta \theta\end{aligned}$$

and

$$\begin{aligned}\Delta L_2 &= I_2 \Delta \omega_2 \cos \Delta \theta + L_1 \Delta \theta \\ &= I_2 \Delta \omega_2 + L_1 \Delta \theta\end{aligned}$$

So the total change in the angular momentum is

$$\begin{aligned}\Delta \vec{L} &= I_1 \Delta \omega_1 \hat{i} + I_2 \Delta \omega_2 \hat{j} - L_2 \Delta \theta \hat{i} + L_1 \Delta \theta \hat{j} \\ &= I_1 \Delta \omega_1 \hat{i} + I_2 \Delta \omega_2 \hat{j} + \Delta \theta \hat{k} \times (L_1 \hat{i} + L_2 \hat{j})\end{aligned}$$

Dividing both sides by Δt and taking proper limit gives

$$\frac{d\vec{L}}{dt} = I_1 \dot{\omega}_1 \hat{i} + I_2 \dot{\omega}_2 \hat{j} + \vec{\omega} \times \vec{L}$$

This gives you some idea about where this equation comes from. Of course in a more accurate treatment, rotations about the other axes also have to be taken into account. For infinitesimal rotations, they can all be added up and give the general equation

$$\frac{d\vec{L}}{dt} = (I_1\dot{\omega}_1\hat{i} + I_2\dot{\omega}_2\hat{j} + I_3\dot{\omega}_3\hat{k}) + \vec{\omega} \times \vec{L}$$

This gives

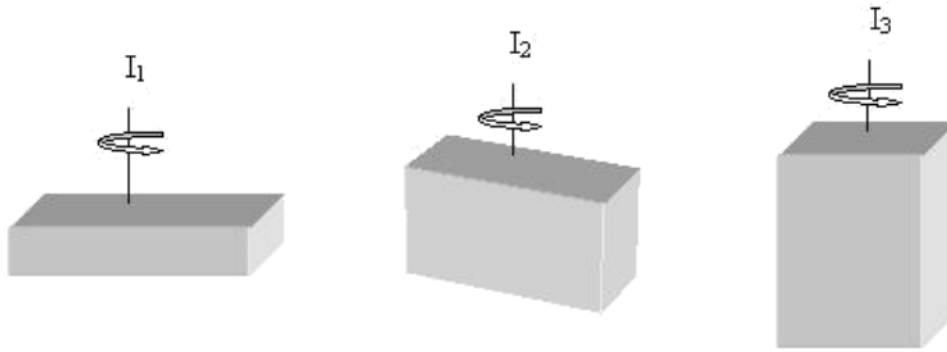
$$\begin{aligned}\dot{L}_1 &= I_1\dot{\omega}_1 + (\vec{\omega} \times \vec{L})_1 \\ &= I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) \\ \dot{L}_2 &= I_2\dot{\omega}_2 + (\vec{\omega} \times \vec{L})_2 \\ &= I_2\dot{\omega}_2 + \omega_3\omega_1(I_1 - I_3) \\ \dot{L}_3 &= I_3\dot{\omega}_3 + (\vec{\omega} \times \vec{L})_3 \\ &= I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1)\end{aligned}$$

Each one of these rates of change should be equal to the component of the torque in that direction .Thus

$$\begin{aligned}\tau_1 &= I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) \\ \tau_2 &= I_2\dot{\omega}_2 + \omega_3\omega_1(I_1 - I_3) \\ \tau_3 &= I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1)\end{aligned}$$

These are the most general equations governing the dynamics of a rigid body and are known as Euler's equations. I now use it to explain the third experiment I had suggested in the beginning of Lecture 21.

Example 4: Hold a rectangular box at a height with one of its faces perpendicular to the vertical, give it a spin and let it drop (see figure 11). Describe its subsequent rotational motion.



A box being given spin about different axes and being dropped from a height. Its moments of inertia about three different axes are shown.

Figure 11

This is an example of torque-free ($\vec{\tau} = 0$) motion because there is no torque on the box about its centre of mass. Thus its rotational motion is governed by the equations

$$\begin{aligned} I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2) &= 0 \\ I_2 \dot{\omega}_2 + \omega_3 \omega_1 (I_1 - I_2) &= 0 \\ I_3 \dot{\omega}_3 + \omega_1 \omega_2 (I_2 - I_1) &= 0 \end{aligned}$$

For a box similar to the one shown in figure 11 we would generally have $I_3 > I_2 > I_1$.

Let me first consider the case when the box is given a spin about its principal axis 1. Let me also assume that in the process I also disturb it and give it very small angular velocities ω_2 and ω_3 about its axes 2 and 3, respectively. Since both ω_2 and ω_3 are very small, their product is second-order in smallness and will be ignored. The Euler equations and there are then as given below.

$$\begin{aligned} I_1 \dot{\omega}_1 &= 0 & \dots (I) \\ I_2 \dot{\omega}_2 + \omega_3 \omega_1 (I_1 - I_3) &= 0 & \dots (II) \\ I_3 \dot{\omega}_3 + \omega_1 \omega_2 (I_2 - I_1) &= 0 & \dots (III) \end{aligned}$$

The first equation implies that ω_1 is a constant. Let me call it the spin rate ω_0 . Using this fact the other two equations are dealt with as follows. Differentiate equation (II) with respect to time to get

$$I_2 \ddot{\omega}_2 + \omega_0 \dot{\omega}_3 (I_1 - I_3) = 0$$

and substitute for $\dot{\omega}_3$ from equation (III) to obtain

$$\ddot{\omega}_2 - \frac{(I_1 - I_3)(I_2 - I_1)\omega_0^2}{I_2 I_3} \omega_2 = 0$$

Since $I_3 > I_2 > I_1$, the equation above is of the form

$$\ddot{\omega}_2 + \Omega^2 \omega_2 = 0 \quad \text{where} \quad \Omega^2 = \frac{(I_3 - I_1)(I_2 - I_1)}{I_2 I_3} \omega_0^2$$

Its solution is of the form

$$\omega_2 = A \sin \Omega t + B \cos \Omega t$$

One can similarly get equation for ω_3 also and see that it also has similar oscillatory solution. This implies that as the box falls down it spins about axis 1 and oscillates about axes 2 and 3. Since magnitudes of ω_2 and ω_3 are small, you see the box fall essentially spinning only. The same thing will happen if we give initial spin about axis 3. However something different happens when the initial spin is about axis 2. Assuming ω_1 and ω_3 to be small, in this case the Euler equations take the following form right after the release of the box.

$$\begin{aligned} \ddot{\omega}_1 - \frac{(I_2 - I_1)(I_3 - I_2)\omega_0^2}{I_1 I_3} \omega_1 &= 0 \\ \ddot{\omega}_2 &= 0 \\ \ddot{\omega}_3 - \frac{(I_2 - I_1)(I_3 - I_2)\omega_0^2}{I_1 I_3} \omega_3 &= 0 \end{aligned}$$

The second equation above implies that ω_2 is a constant and with $I_3 > I_2 > I_1$, the other two equations take the form

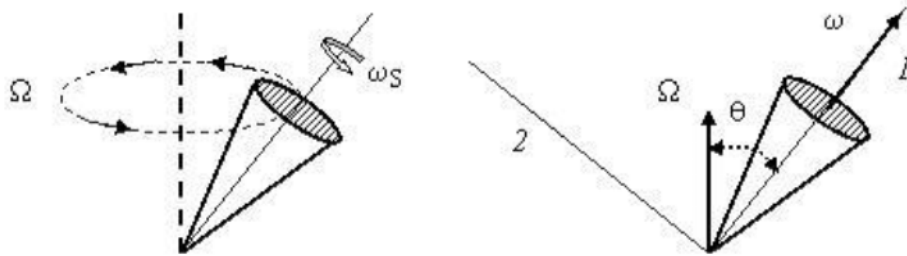
$$\ddot{\omega}_1 - \Omega^2 \omega_1 = 0, \quad \ddot{\omega}_3 - \Omega^2 \omega_3 = 0 \quad \text{where} \quad \Omega^2 = \frac{(I_2 - I_1)(I_3 - I_2)}{I_1 I_3} \omega_0^2$$

Solution of these equations is of the form

$$A \exp(\Omega t) + B \exp(-\Omega t)$$

which indicates that right after the release, the angular velocities about axes 1 and 3 will grow very fast and take on a large value. Thus the box will start rotating about all three axes and that is what you observe. Thus we see that a rigid body is stable when it is given a spin about the axes having the smallest or the largest moment of inertia. However, if given a spin about the axis with intermediate moment of inertia, it will be unstable. Next I take up the case of precessing top that I had not solved by employing Euler's equations earlier. This is an example where a torque is also being applied on the system

Example 5: Apply Euler's equations to a precessing top and get its precession frequency Ω . The top has a mass m and is spinning at a rate of ω_S (see figure 12). Its centre of gravity is at a distance l from the pivot point.



A precessing top (left) and its principal axes and spin and precession angular velocities at time $t=0$ (right).

Figure 12

I have already discussed about the principal axes of the top in example 2 above.

With $I_1 = I, I_2 = I_3 = I_\perp$ the Euler's equations for the top are

$$\begin{aligned}\tau_1 &= I_1 \dot{\omega}_1 \\ \tau_2 &= I_\perp \dot{\omega}_2 + \omega_3 \omega_1 (I - I_\perp) \\ \tau_3 &= I_\perp \dot{\omega}_3 + \omega_1 \omega_2 (I_\perp - I)\end{aligned}$$

Now in applying Euler's equations you have to keep in mind that the top is spinning. As such its principle axes 2 and 3 also rotate about axis 1 with angular frequency ω_S . So the components of angular frequency and torque in the direction of these axes also change with time. Taking time at which the position of the top is shown in figure 12 to be $t = 0$, I draw in figure 13 the position of axes 2 and 3 at time t . In this figure, I have neglected the angle $\omega_S t$ through which the top and therefore the torque vector itself has rotated. In other words I have assumed that $\omega_S \gg \Omega$. Thus the angular velocity and torque are shown where they were at $t = 0$.

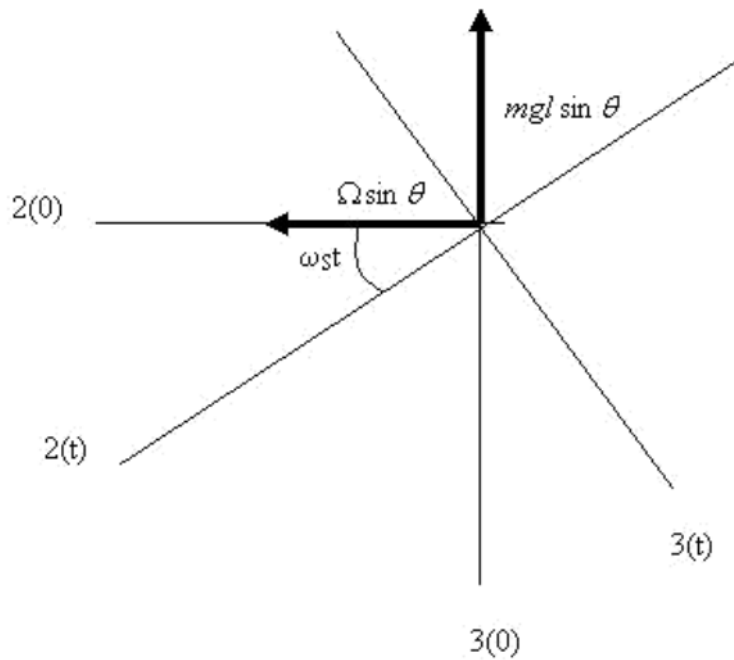


Figure 13

Looking at figure 13, it is clear that the components of the angular velocity and the torque are

$$\omega_1 = \omega_s + \Omega \cos \theta \approx \omega_s \quad \omega_2 = \Omega \sin \theta \cos(\omega_s t) \quad \omega_3 = -\Omega \sin \theta \sin(\omega_s t)$$

$$\tau_1 = 0 \quad \tau_2 = -mgl \sin \theta \sin(\omega_s t) \quad \tau_3 = -mgl \sin \theta \cos(\omega_s t)$$

Substituting these in the Euler's equation for the top gives

$$\begin{aligned} \dot{\omega}_1 &= 0 \\ -mgl \sin \theta \sin(\omega_s t) &= -I_{\perp} \Omega \omega_s \sin \theta \sin(\omega_s t) - (I - I_{\perp}) \Omega \omega_s \sin \theta \sin(\omega_s t) \\ -mgl \sin \theta \cos(\omega_s t) &= -I_{\perp} \Omega \omega_s \sin \theta \cos(\omega_s t) + (I_{\perp} - I) \Omega \omega_s \sin \theta \cos(\omega_s t) \end{aligned}$$

The first of these equations gives $\omega_1 = \text{constant} = \omega_s$. The other two equations give the same answer which is

$$\Omega = \frac{mgl}{I \omega_s}$$

This is the answer that we have seen earlier. In solving the Euler's equations for the top, we made the assumption of $\omega_s \gg \Omega$. Further we assumed that the top only precesses about the vertical.

However, there is no reason why it cannot possess a horizontal angular velocity Ω_H also.

Assuming the existence of Ω and Ω_H and then solving the Euler's equations will give a more complete solution for the motion of a spinning top. It in fact gives the nutating motion also. You may want to try getting this general solution.

With this lecture I end of the topic of rigid-body rotation.

Harmonic oscillator I: Introduction

Having analyzed the motion of particles in different situations, let us now focus on a very special kind of motion: that of oscillations. This is a very general kind of motion seen around you: A particle moving around the bottom of a cup, a pendulum swinging, a clamped rod vibrating about its equilibrium position or a string vibrating. A good first approximation to these motions is the simple harmonic oscillation. Let us see what does that mean? At a stable equilibrium point, the force on a body is zero; not only that, as a particle moves away from equilibrium, its potential energy increases and it is pulled back towards the equilibrium point. Thus around a stable equilibrium point x_0 (for simplicity, let me take one-dimensional motion) the potential energy

$\phi(x)$ can be written as

$$\phi(x_0 + \Delta x) = \phi(x_0) + \left. \frac{d\phi}{dx} \right|_{x_0} \Delta x + \frac{1}{2} \left. \frac{d^2\phi}{dx^2} \right|_{x_0} \Delta x^2 + \dots$$

Since at an equilibrium point, the force $F(x_0)$ on the particle vanishes,

$$\left. \frac{d\phi}{dx} \right|_{x_0} = -F(x_0) = 0$$

Further, because $\phi(x)$ has a minimum at x_0 , this gives

$$\phi(x_0 + \Delta x) = \phi(x_0) + \frac{1}{2} k \Delta x^2 \quad \text{with} \quad k = \left. \frac{d^2\phi}{dx^2} \right|_{x_0} > 0$$

Writing $\Delta x = y$ I get

$$\phi(y) = \phi(x_0) + \frac{1}{2} k y^2$$

and the corresponding equation of motion for a mass m as

$$m\ddot{y} = -ky \quad \text{or} \quad m\ddot{y} + ky = 0$$

As I will show a little later, the solution of this equation is of the form

$$y(t) = A \sin \omega t + B \cos \omega t$$

and is known as the simple harmonic motion. It is the simplest possible motion about a stable equilibrium point. Of course if $k = 0$, the force will have higher order dependence on y and the motion becomes more complicated. Further, even if $k \neq 0$, if we include higher order terms, the resulting motion will become more complex. It is for this reason that we call the motion above simple harmonic motion. We will see that this itself is quite a rich system. A system that performs simple harmonic motion is called a simple harmonic oscillator. A prototype of this system is the spring-mass system with k being the spring constant and m the mass of the block on the spring (figure 1).

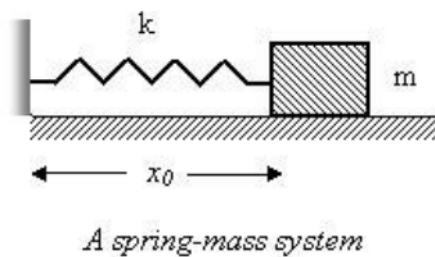


Figure 1

In these lectures, I will talk about the motion of this system and how it is represented by a phasor diagram. I will then introduce damping into the system. The simplest damping is a constant opposing force like friction and next level is a damping proportional to the velocity. Then I will apply a force on the system and see the motion of force damped and undamped oscillator. Along the way, I will solve many examples to show wide applicability of simple harmonic motion.

To start with let us take our prototype system of mass and spring with unstretched length of the spring x_0 so that equilibrium distance of the mass is x_0 . Now when the mass is displaced about x_0 by x in the positive direction, the force is in negative direction so that

$$m\ddot{x} = -kx$$

or

$$\ddot{x} + \omega_0^2 x = 0 \quad \text{with} \quad \omega_0^2 = \frac{k}{m}$$

This is the general equation for simple harmonic oscillator. Recall that in such cases we assume a solution of the form

$$x = e^{\lambda t}$$

and substitute it in the equation to get

$$\lambda^2 e^{\lambda t} + \omega_0^2 e^{\lambda t} = 0$$

Since this equation is true for all times, we should have

$$\lambda^2 + \omega_0^2 = 0 \Rightarrow \lambda = \pm i \omega_0$$

Thus there are two solutions $e^{i\omega_0 t}$ and $e^{-i\omega_0 t}$. A general solution is then given in terms of a linear combination of the two solutions so let us write

$$x(t) = A e^{i\omega_0 t} + B e^{-i\omega_0 t}$$

Since $x(t)$ is real it is clear that $B = A^*$. Thus

$$x(t) = A e^{i\omega_0 t} + A^* e^{-i\omega_0 t}$$

If we take $A = A_R + iA_I$, where both A_R and A_I are real then the solution above takes the form

$$x(t) = 2A_R \cos \omega_0 t - 2A_I \sin \omega_0 t$$

which alternatively can be written as

$$C \cos \omega_0 t + D \sin \omega_0 t$$

Another equivalent way of writing the solution is

$$x(t) = A \sin(\omega_0 t + \phi) \quad \text{or} \quad x(t) = A \cos(\omega_0 t + \phi)$$

where

$$A = \sqrt{C^2 + D^2} ; \sin \phi = \frac{C}{\sqrt{C^2 + D^2}} \text{ and } \cos \phi = \frac{D}{\sqrt{C^2 + D^2}}$$

A is the maximum distance that the mass travels during a simple harmonic oscillation. It is known as the amplitude of oscillation. The quantity $(\omega_0 t + \phi)$ is known as the phase with ϕ being the initial phase. All the boxed equations above are equivalent ways of writing the solution for a

harmonic oscillator. The general graph depicting the solution $x(t) = A \sin(\omega_0 t + \phi)$ is given in figure 2.

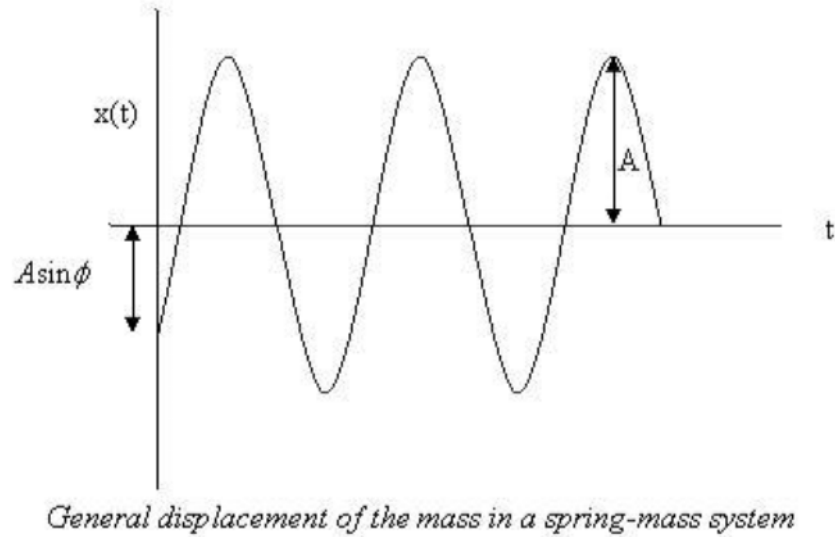


Figure 2

Thus A is the maximum distance traveled by the block and $A \sin \phi$ gives its initial displacement. The constants C and D or A and ϕ are determined by the initial conditions, i.e. initial displacement and velocity of the mass. In general any two conditions are enough to determine the constants.

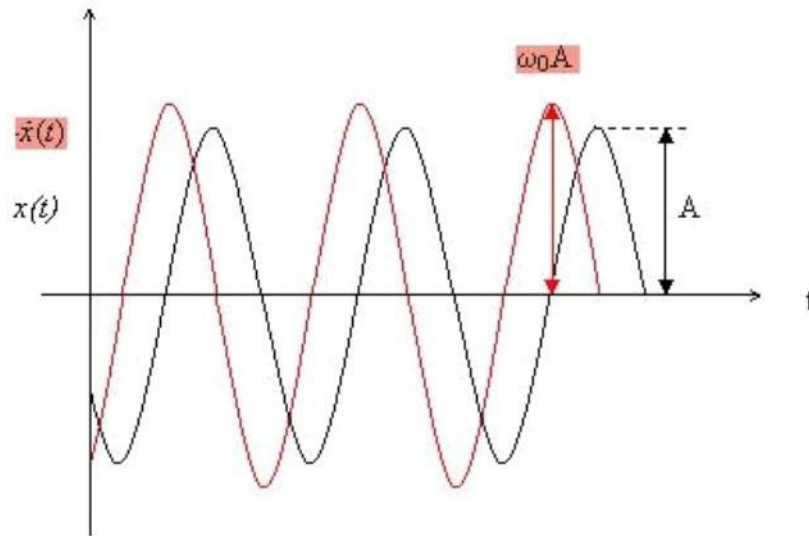
For a displacement

$$\begin{aligned} x(t) &= C \cos \omega_0 t + D \sin \omega_0 t \\ &= A \sin(\omega_0 t + \phi) \end{aligned}$$

the velocity of the mass is given by

$$\begin{aligned} v(t) &= \dot{x}(t) \\ &= \omega_0 (C \cos \omega_0 t - D \sin \omega_0 t) \\ &= \omega_0 A \cos(\omega_0 t + \phi) \end{aligned}$$

Thus the maximum possible magnitude of the velocity is $\omega_0 A$. The general displacement and the corresponding velocity of the mass with respect to time are displayed in figure 3.



Displacement and velocity in a harmonic oscillator

Figure 3

It is clear from the figure that for a given displacement, the velocity is such that when displacement is at its maximum or minimum, the velocity is zero and when the displacement is zero, the velocity has the largest magnitude. This is physically clear. When the spring is compressed or stretched to its maximum, the particle is at rest and when the particle passes through the equilibrium point, its speed is at its maximum. Let me now solve a few examples.

Example 1: In a spring-mass system $k = 16 \text{ N/m}$ and $m = 1 \text{ kg}$. If the mass is displaced by $.05 \text{ m}$ and released from rest, find its subsequent motion.

$$x(t) = C \sin \omega_0 t + D \cos \omega_0 t$$

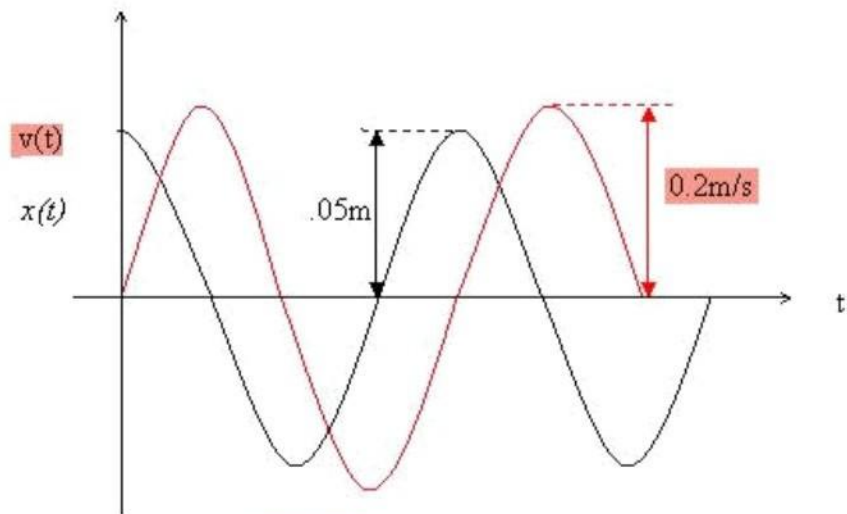
$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{16} = 4 \text{ rad/s}$$

Using the initial conditions I get

$$x(0) = D = 0.05 \text{ m}$$

$$\dot{x}(0) = \omega_0 C = 0 \Rightarrow C = 0$$

So the solution is $x(t) = .05 \cos 4t$ with the maximum speed of 0.2 m/s . The solution $x(t)$ is plotted in figure 4. Also plotted there is the velocity $v(t)$ of the mass as it performs its motion. Notice that from the $x(t)$ curve, the velocity can be easily plotted by taking its slope.



Displacement and velocity in example 1 with the initial condition $x(0)=0.5\text{m}$ and $v(0)=0$.

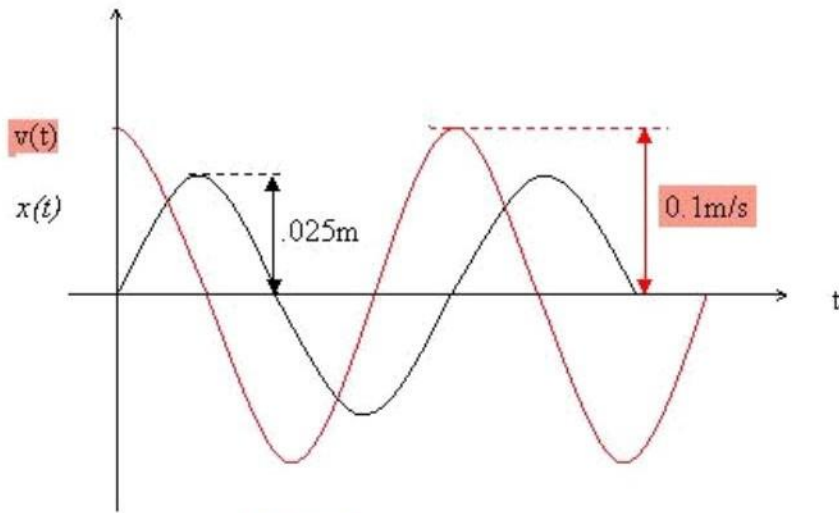
Figure 4

Let me now show you how the solution changes when the initial conditions are different. Suppose instead of pulling the mass and releasing it, I give it an initial velocity of $.1\text{m/s}$ toward the right from the equilibrium. In that case

$$x(0) = D = 0$$

$$\dot{x}(0) = \omega_0 C = 0.1 \Rightarrow C = 0.025$$

So $x(t) = 0.025 \sin 4t$. Obviously the maximum speed in this case is 0.1m/s , that given in the beginning. The solution looks like shown in figure 5.



Displacement and velocity in example 1 with the initial condition $x(0)=0$ and $v(0)=0.1\text{m/s}$.

Figure 5

Third possibility of initial conditions is when I take the mass to a displacement of .05m and push it towards the equilibrium point with a speed of .1m/sec. Then

$$x(0) = D = .05$$

$$\dot{x}(0) = \omega_0 C = -.1 \Rightarrow C = -.1/4 = -.025$$

Thus the solution is $x(t) = -0.025 \sin \omega_0 t + 0.05 \cos \omega_0 t$. If we wish to express this as $x(t) = A \sin (\omega_0 t + \phi)$ then

$$A = \sqrt{(.05)^2 + (.025)^2} = .056$$

and

$$\sin \phi = \frac{.05}{\sqrt{(.05)^2 + (.025)^2}} ; \cos \phi = \frac{-.025}{\sqrt{(.05)^2 + (.025)^2}}$$

$$\text{and } \tan \phi = -2 \text{ with } \frac{\pi}{2} < \phi < \pi$$

This gives $\phi \approx 116^\circ$ and $x(t) = .056 \sin(4t + 116^\circ)$. The maximum speed in this case is $v_{\max} = 4 \times 0.056 = 0.224\text{m/s}$. So the graph of the motion looks like that shown in figure 6.

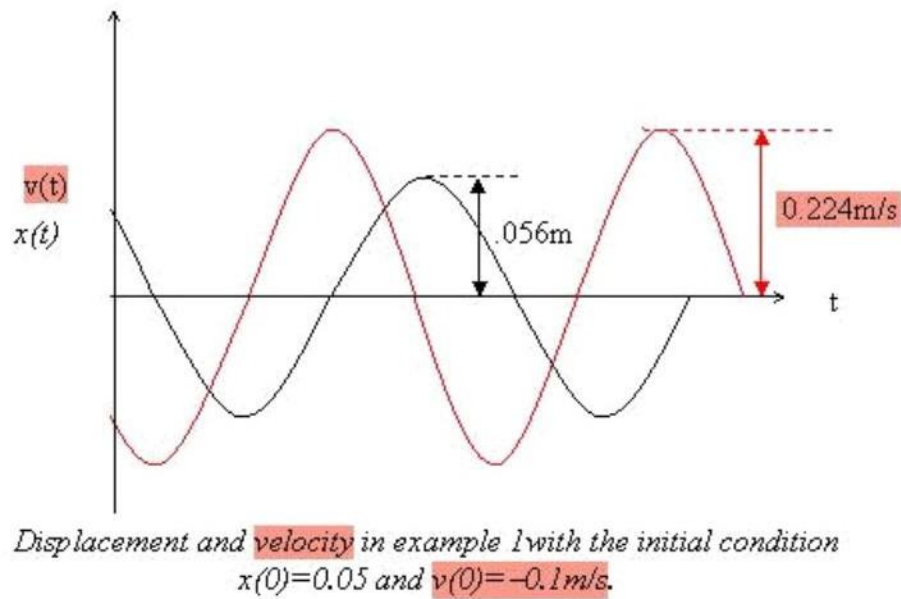
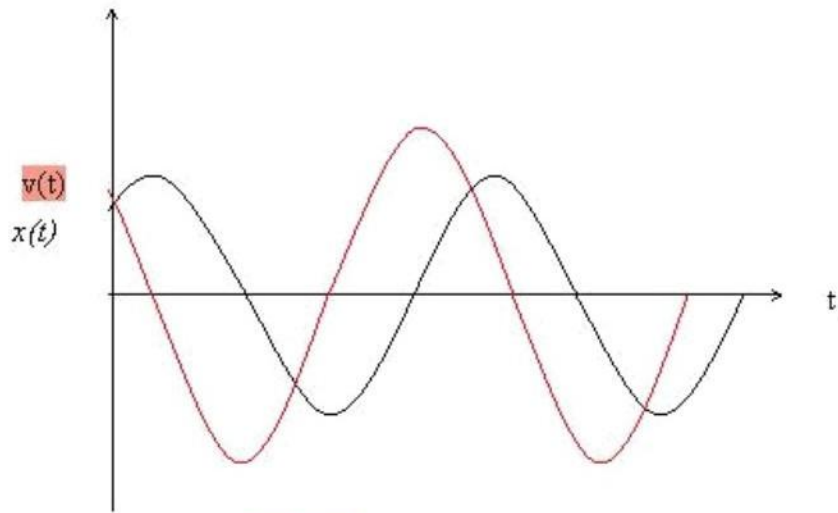


Figure 6

From the graph it is very clear that initially the speed of the particle increases in the negative direction and then the particle starts slowing down, stopping at the full compression of the spring, as is clear from the plot of its displacement.

If in the case studied just now, the mass was thrown out instead of being pushed in, it would have a positive velocity to start with but the speed would be decreasing at that moment. Then the mass will travel out to its maximum displacement and would then turn back. The general plot of displacement and velocity versus time would then look as in figure 7. I will leave it for you to work out the numbers for amplitude and initial phase.



Displacement and velocity in example 1 with the initial condition $x(0)=0.05$ and $v(0)=+0.1\text{ m/s}$.

Figure 7

Example 2: In the second example I show that about any stable equilibrium point, the motion to a good degree is simple harmonic. let us take two charges of $10\text{ }\mu\text{C}$ each at a distance of half a meter so that is a positive charge of $5\text{ }\mu\text{C}$ is kept at the centre, its experiences no force (see figure 8). The $5\text{ }\mu\text{C}$ charge is confined to move along the line joining the two charges. If displaced by a small distance from its equilibrium position, what kind of motion does it perform?

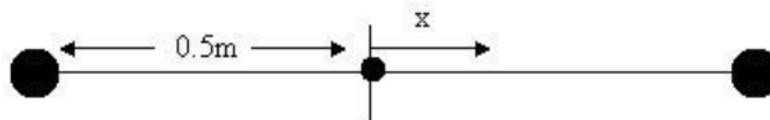


Figure 8

When the $5\text{ }\mu\text{C}$ is displaced to the right by x , the force on it is

$$F = 9 \times 10^9 \times \left(\frac{50 \times 10^{-12}}{(0.5 + x)^2} - \frac{50 \times 10^{-12}}{(0.5 - x)^2} \right)$$

$$\cong -3.6x$$

In obtaining the force above, we have used the binomial theorem to expand $(0.5 \pm x)^{-2}$. Since the force is proportional to the displacement and in direction opposite to it, the charge will perform simple harmonic motion.

Let me now look at some other examples, going beyond the spring-mass system.

Example 3: A disc of mass M and radius R is hanging on a wall about a point on its periphery (see figure 9). If it is displaced from its initial position by small angle θ_0 and released, find its subsequent motion.

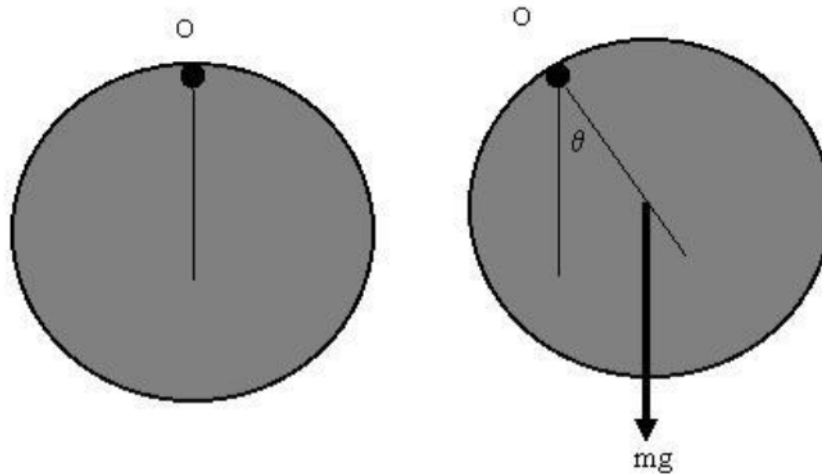


Figure 9

This is a case where a rigid body is moving under distributed forces so we use angular momentum to describe its motion. The equation of its motion therefore is

$$\begin{aligned} I_o \dot{\omega} &= I_o \ddot{\theta} = -MgR \sin \theta \\ &= -MgR \theta \text{ (small angle)} \end{aligned}$$

By transformation theorem,

$$I_o = I_{\text{about CM}} + MR^2 = \frac{3MR^2}{2}$$

So the equation of motion becomes

$$\ddot{\theta} + \frac{2g}{3R} \theta = 0$$

This means that in general the motion of the disc would be simple harmonic and will be given as

$$\theta(t) = C \sin \omega_0 t + D \cos \omega_0 t \quad \text{with} \quad \omega_0 = \sqrt{\frac{2g}{3R}}$$

The initial conditions in this case give $C = 0$ and $D = \theta_0$. Therefore the solution in the present

case is
$$\theta(t) = \theta_0 \cos \sqrt{\frac{2g}{3R}} t.$$

Example 4: As the final example here, let me take a particle moving in a potential

$$U(x) = \frac{A}{x^2} + Bx^2 \quad (A > 0, B > 0)$$
. The potential has a minimum at x_0 given by

$$\left. \frac{d}{dx} \left(\frac{A}{x^2} + Bx^2 \right) \right|_{x_0} = 0$$

$$\Rightarrow -\frac{2A}{x_0^3} + 2Bx_0 = 0$$

or

$$x_0 = \left(\frac{A}{B} \right)^{\frac{1}{4}}$$

You can yourself check that the second derivative at this point is positive and its value is $8B$. For very small displacements x about this point we have the change in the potential energy given as

$$\Delta U(x) = \frac{A}{(x_0 + x)^2} + B(x_0 + x)^2$$

which by binomial theorem or the Taylor series expansion leads to

$$\Delta U(x) = U(x_0) + \frac{1}{2}(8B)x^2$$

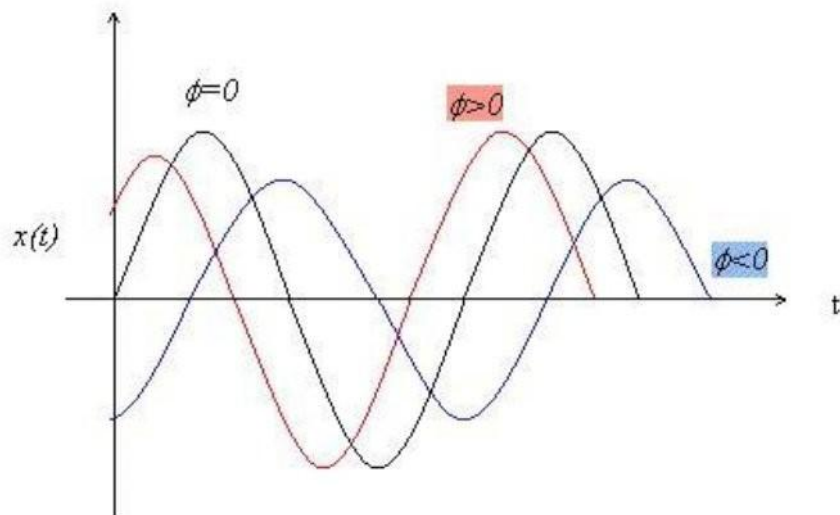
This gives an equivalent spring constant of $k=8B$ and frequency of oscillation
$$\omega_0 = \sqrt{\frac{8B}{m}}.$$

Having solved these examples I now wish to discuss a very important topic of phase and phase difference in a simple-harmonic motion. I will spend some time discussion phasor diagrams give a feel for the phase.

Phase and Phase difference in simple harmonic motion : In general the solution of a simple harmonic equation is

$$x(t) = A \sin(\omega t + \phi)$$

As mentioned earlier A is known as the amplitude and $(\omega t + \phi)$ as the phase. ϕ is a constant depending on the initial conditions and we call it the phase constant. Let us now see how does the motion look for different values of the phase constant ϕ . The displacement versus time plots for different signs of the phase constant are shown in figure 10.



*Displacement versus time graphs for different values of phase constant.
Black: $\phi=0$; Red: $\phi>0$ and blue is $\phi<0$.*

Figure 10

For $\Phi > 0$ the motion at $t = 0$ begin at a value or phase angle that it would have slightly later in the $\phi = 0$ case. On the other hand, for $\Phi < 0$ the motion is such that a particular displacement for the $\phi = 0$ case is reached at a later time. The motion lags behind the $\phi = 0$ motion. I leave it for you to figure out yourself how the corresponding velocities are related.

Let us now at the special case of $\phi = 180^\circ$. In this case I get

$$x(t) = A \sin(\omega t + 180^\circ) = -A \sin \omega t$$

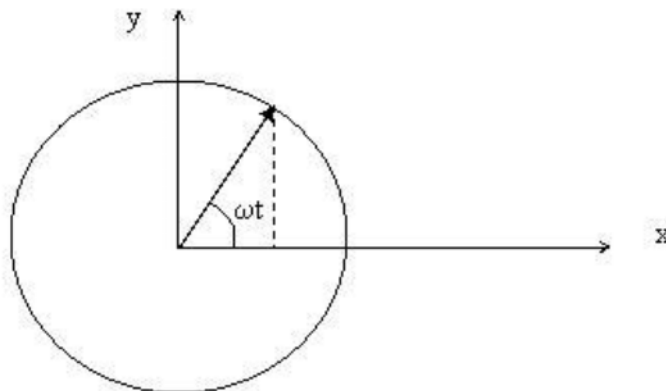
and for $\phi = -180^\circ$

$$x(t) = A \sin(\omega t - 180^\circ) = -A \sin \omega t$$

So you see that a phase difference of 180° , whether positive or negative, means the same thing. I would like you to plot the displacement versus time graph for these particular cases. For the phases in between you should see for yourself how the displacements at $t = 0$ are different from $\phi = 0$ case.

A good way of visualizing the simple harmonic motion is the phasor or vector diagram. I discuss that next.

Phasor or vector diagram: A nice geometric way of looking at various quantities in a simple harmonic motion is the vector or a phasor diagram. You may have seen it in your 12th grade while studying AC circuits. Let me show you how we represent $x(t) = A \cos \omega t$ in a geometric way. You see that displacement in this case is the x component of a vector making an angle ωt from the x -axis. Thus the displacement is represented as shown in figure 11. The motion described by $x(t) = A \cos \omega t$ is thus given by the projection of a vector of length A , rotating counterclockwise at a rate ω , on the x -axis.



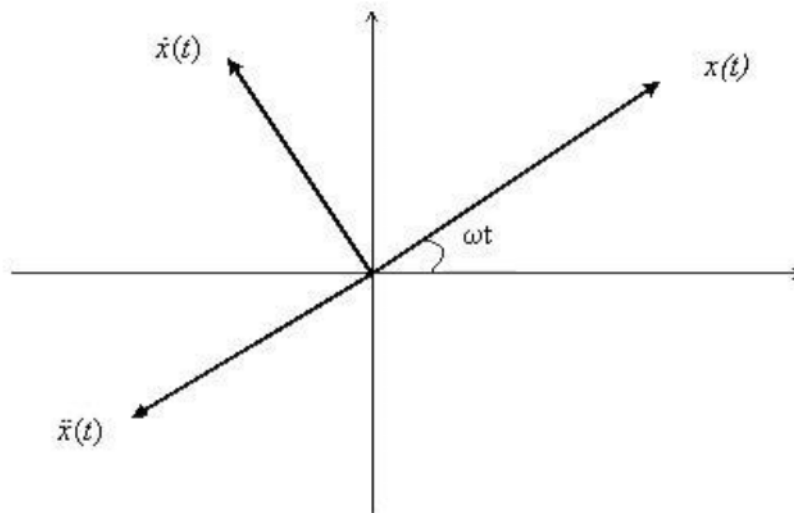
Projection of a vector \vec{A} rotating counterclockwise at a rate ω on the x -axis gives $x(t) = A \cos \omega t$

Figure 11

Let us now see how the velocity $\dot{x}(t)$ and the acceleration will be represented in this scheme? The velocity and acceleration are given as

$$\begin{aligned}\dot{x}(t) &= -\omega A \sin \omega t = \omega A \cos \left(\omega t + \frac{\pi}{2} \right) \\ \ddot{x}(t) &= -\omega^2 A \cos \omega t = \omega^2 A \cos(\omega t + \pi)\end{aligned}$$

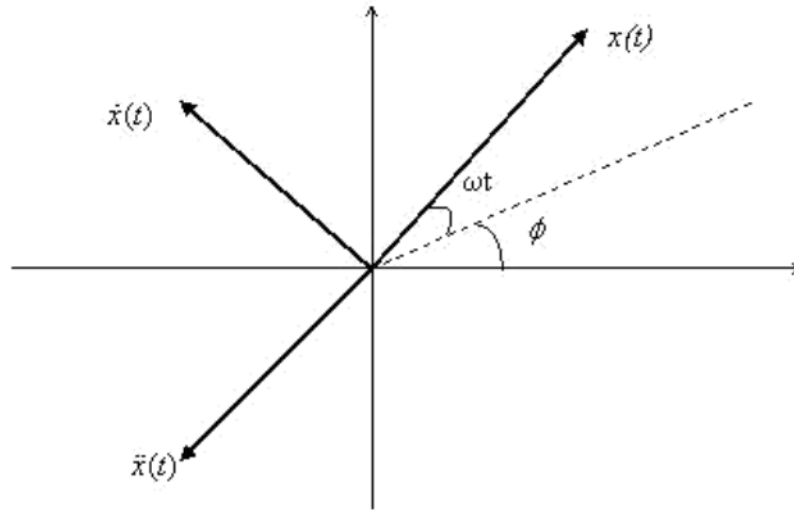
The displacement, velocity and acceleration are shown in the phasor diagram in figure 12. A general feature that we observe from this phase diagram is that the velocity vector is always $\frac{\pi}{2}$ ahead (measuring counterclockwise) of the displacement vector and the acceleration vector is at π (ahead or behind?) the displacement.



Displacement, velocity and acceleration for a simple harmonic motion shown on a phasor diagram

Figure 12

So far we have discussed the simple case of $x(t) = A \cos \omega t$. What about the general case of $x(t) = A(\cos \omega t + \phi)$. This is also equally simple. All we have to do is keep the initial position of the vector at $t = 0$ at an angle ϕ from the x-axis and start rotating it from there. The velocity vector and the acceleration vector are then going to be given at $\frac{\pi}{2}$ and π from it, as discussed above. This is shown in figure 13.



Displacement $x(t) = A(\cos \omega t + \phi)$, and the corresponding velocity and acceleration for a simple harmonic motion

Figure 13

Recall that in the middle of this lecture I had solved a spring-mass problem with different initial conditions. I would like you to make the phasor diagram to represent the motion of the mass in many different situations like those considered above. Do not solve for $x(t)$ to start with, just make the phasor diagram directly to see if you have got a feel for motion under different conditions.

Finally in this lecture I look at the energy of a system performing simple harmonic motion. The potential energy $U(x)$ and the kinetic energies T are

$$U(x) = \frac{1}{2} kx^2 \quad T = \frac{1}{2} m\dot{x}^2$$

The total energy E is of course a sum of the two. With $x(t) = A \cos(\omega t + \phi)$ this gives

$$E = \frac{1}{2} kA^2 \cos^2(\omega t + \phi) + \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \phi)$$

Since $\omega^2 = \frac{k}{m}$, we get

$$E = \frac{1}{2} kA^2$$

Thus the energy depends on the square of the amplitude. This makes sense because if I stretch a

spring by A , the energy stored in it is $\frac{1}{2}kA^2$. On releasing the mass it performs oscillations of amplitude A . Thus you see that amplitude A immediately implies a total energy given above.

I have now set up all the basic concepts of simple harmonic motion. In the coming lectures I will introduce damping in the system and see how it evolves.

Harmonic oscillator II: damped oscillator

In the previous lecture, I covered some basic aspects of simple harmonic oscillations. We considered the equation

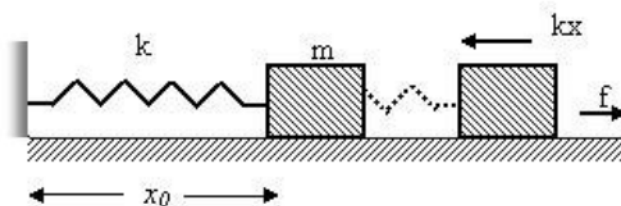
$$m\ddot{x} + kx = 0$$

and saw how its motion is described. A general solution of this equation is

$$x(t) = A \cos(\omega_0 t + \phi) \text{ with } \omega_0 = \sqrt{\frac{k}{m}}$$

I now make the system little more realistic and introduce damping into the system. Let us first look at what happens if we introduce friction into the system. I consider again our prototype spring-mass system and let there be a constant frictional force f on the mass. This force will always oppose the motion so the system will eventually come to a stop. Let us see when does it do that?

The simplest way of seeing when the system will stop is through the consideration of energy. But I would like to solve the problem by employing the equation of motion. I will later solve it from energy considerations also. Here is one case where I will have to analyze motion step by step because as the velocity direction changes, so does the force direction. So let us pull the spring out to a distance A and let it move towards the equilibrium point (see figure 1).



A spring-mass system with friction. Force applied by the spring and frictional force are both shown.

Figure 1

When the block is moving towards the left, equation governing its motion will be

$$m\ddot{x} = -kx + f$$

In the above, the frictional force f sign is positive because the mass is moving in the negative x direction and therefore the frictional force is in positive x direction. This equation can be recast into the form

$$m\ddot{x} + \omega_0^2 x = \frac{f}{m} \quad \text{with} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

We have encountered such kind of equation earlier. It has a homogeneous part $\ddot{x} + \omega_0^2 x = 0$ and an inhomogeneous term on the right-hand side. So the general solution is

$$x = x_{\text{homogeneous}} + x_{\text{inhomogeneous}}$$

where

$$x_{\text{homogeneous}} = C \cos \omega_0 t + D \sin \omega_0 t$$

$$x_{\text{inhomogeneous}} = \frac{f}{(m \omega_0^2)} = \frac{f}{k}$$

Thus

$$x(t) = C \cos \omega_0 t + D \sin \omega_0 t + \frac{f}{k}$$

With the initial conditions $x(0) = A$ and $\dot{x}(0) = 0$, the solution is

$$x(t) = \left(A - \frac{f}{k} \right) \cos \omega_0 t + \frac{f}{k}$$

This is the solution when the block is moving to the left. Since

$$\dot{x} = -\left(A - \frac{f}{k} \right) \omega_0 \sin \omega_0 t$$

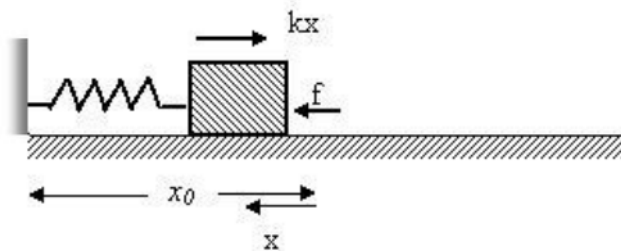
so the block will come to a stop when $(\omega_0 t) = \pi$. At that time

$$\begin{aligned}
 x(t) &= -A + \frac{f}{k} \omega_0 \cos \omega_0 t + \frac{f}{k} \\
 &= -A + \left(\frac{2f}{k} \right)
 \end{aligned}$$

So by the time the block comes to a stop it has lost $\frac{2f}{k}$ distance from its amplitude. And this loss is irrespective of the distance from where the block starts its motion from. This should then also happen when the block starts coming back. Let us find that out. On its way back (see figure 2), the block follows the equation

$$m\ddot{x} = -kx - f$$

Notice that the sign of the friction force is now negative. This is because now the block is traveling to the right and therefore the friction force acts towards the left (see figure 2).



A spring-mass system with friction when the spring is compressed. Force applied by the spring and frictional force are both shown.

Figure 2

Now we have to solve this equation with the initial condition that

$$x(0) = -A + \left(\frac{2f}{k} \right) \quad \text{and} \quad \dot{x}(0) = 0$$

I leave it as an exercise for you to get the solution. It is

$$x(t) = -\left(A - \frac{3f}{k} \right) \cos \omega_0 t - \frac{f}{k}$$

The corresponding velocity is proportional to $\omega_0 t$, and therefore goes to zero again after a time interval of $\frac{\pi}{\omega_0}$. At that time $x = \left(A - \frac{4f}{k} \right)$. Thus every half time the block goes from one

extreme to the other, it loses a distance of $\left(\frac{2f}{k}\right)$, and in each cycle it loses a distance of $\left(\frac{4f}{k}\right)$. Question is how many cycles does the block complete before it comes to a stop. The block stops

when its final displacement is $\frac{f}{k}$. If it completes n cycles before that, we have

$$A - n \frac{4f}{k} = \frac{f}{k} \Rightarrow n = \frac{k}{4f} \left(A - \frac{f}{k} \right)$$

The same result can also be obtained, as I said earlier, by energy methods. If stretched by A the total energy of the system is $\frac{1}{2}kA^2$. Let us say that before stopping, the block it compresses the spring by A_1 . Then its energy will be $\frac{1}{2}kA_1^2$. The loss in the energy is caused by friction. Thus

$$\frac{1}{2}kA^2 - \frac{1}{2}kA_1^2 = \text{energy loss due to friction}$$

The total distance moved by the block is $(A + A_1)$ and so the energy lost against friction is $f(A + A_1)$. Thus the equation transforms to

$$\frac{1}{2}kA^2 - \frac{1}{2}kA_1^2 = f(A + A_1)$$

and gives

$$(A - A_1) = \frac{2f}{k}$$

which is the same loss in amplitude over half a cycle as obtained earlier. The rest of the analysis is the same as done earlier.

Having dealt with the constant friction case, we now consider the most common example of damped oscillations. This is the oscillator where damping force is proportional to the velocity i.e.,

$$\begin{aligned} F_{\text{retardation}} &= -bv \\ &= -b\dot{x} \end{aligned}$$

In this case, the equation of motion is

$$m\ddot{x} = -kx - b\dot{x} \quad \text{or} \quad m\ddot{x} + b\dot{x} + kx = 0$$

Writing $\frac{b}{m} = \gamma$ we get

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0$$

This is the equation for a damped oscillator. The equation is homogeneous in x so we assume a solution $x(t) = e^{\lambda t}$ and substitute it in the equation to get.

$$\lambda^2 + \gamma \lambda + \omega_0^2 = 0$$

which gives

$$\lambda = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

So the general solutions are

$$\exp\left(-\frac{\gamma}{2}t + \sqrt{\frac{\gamma^2}{4} - \omega_0^2}t\right) \quad \text{and} \quad \exp\left(-\frac{\gamma}{2}t - \sqrt{\frac{\gamma^2}{4} - \omega_0^2}t\right)$$

Except in the case when $\frac{\gamma^2}{4} = \omega_0^2$ (we will deal with it later) the behavior of the solution depends on the relation magnitude of γ and ω_0 . Let us first consider the case $\frac{\gamma^2}{4} > \omega_0^2$. In that case

$$\lambda = -\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2} = -\lambda_1 \quad \text{and} \quad \lambda = -\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2} = -\lambda_2$$

The general solution then is

$$x(t) = -Ce^{-\lambda_1 t} + De^{-\lambda_2 t} \quad \text{where} \quad \lambda_2 > \lambda_1$$

This is known as a heavily damped oscillator. The coefficients C and D depend on the initial conditions. For example if I stretch the spring to a distance A and release the block, let us see what happen in this case. By initial conditions

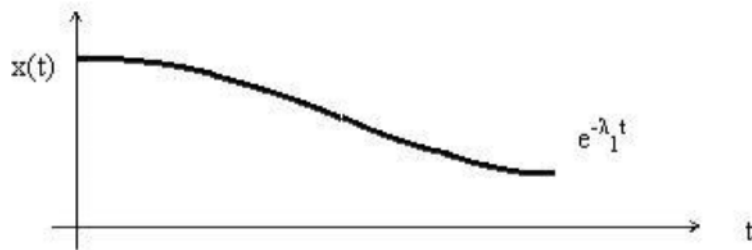
$$x(0) = A = C + D$$

$$\dot{x}(0) = 0 \Rightarrow -\lambda_1 C - \lambda_2 D = 0 \quad \text{or} \quad D = -\frac{\lambda_1}{\lambda_2} C$$

This leads to

$$x(t) = A \left[\frac{\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t}}{\lambda_2 - \lambda_1} \right]$$

As $t \rightarrow \infty$ this solution behaves like $x(t) = \frac{A\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 t}$. The general solution is displayed in figure 3.



Displacement vs. time for a heavily damped oscillator

Figure 3

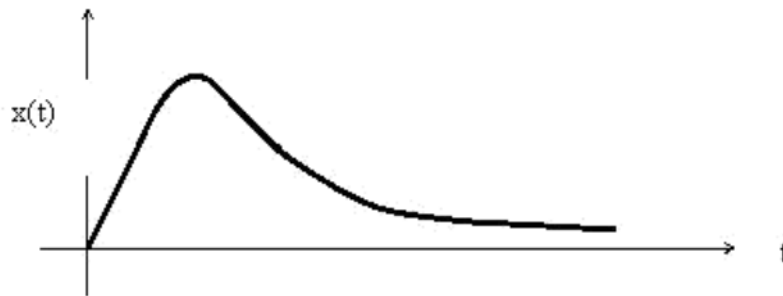
It is clear from the figure that there are no oscillations in this case the block slowly comes to rest at $x = 0$, i.e. the equilibrium point. I now explore another situation. Suppose we give an impulse (speed v) at $t = 0$ then the boundary conditions are

$$C + D = 0 \quad \text{and} \quad -\lambda_1 C - \lambda_2 D = v$$

Thus the solution would be

$$x(t) = \frac{v}{\lambda_2 - \lambda_1} \left(e^{-\lambda_1 t} - e^{-\lambda_2 t} \right)$$

In this case the distance versus time graph looks as shown in figure 4.



Displacement vs. time for a heavily damped oscillator when the block is given an impulse at the equilibrium point.

Figure 4

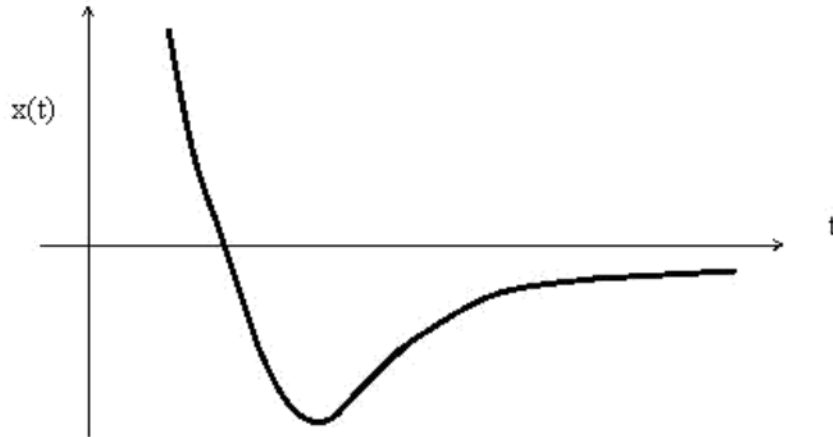
The figure clearly shows that the block goes out to a maximum distance and then comes back and stops at the equilibrium point. So in both the cases studied above the mass does not cross the equilibrium point. Next I ask: what if we stretch the mass out to a distance A and give it an initial impulse from that point (in negative direction). Then the initial conditions will be

$$C + D = A \quad \text{and} \quad -\lambda_1 C - \lambda_2 D = -v$$

Solution in this case comes out to be

$$x(t) = -\frac{v}{\lambda_2 - \lambda_1} \left(e^{-\lambda_1 t} - e^{-\lambda_2 t} \right) + \frac{A}{\lambda_2 - \lambda_1} \left(\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t} \right)$$

The solution is plotted schematically in figure 5.



Displacement vs. time for a heavily damped oscillator when the block is given an impulse after stretching the spring.

Figure 5

It is clear that in this case the particle moves towards the equilibrium point, crosses it, goes a distance and comes back. However on its way back it slowly comes to rest at the equilibrium point and does not cross it. So in heavy damping cases, the block passes the equilibrium point at most once and its distance decays exponentially as $e^{-\lambda_1 t}$.

To summarize, I have covered three cases for the heavy damping situation and got

$$x(t) = \frac{A}{\lambda_2 - \lambda_1} [\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t}]$$

(i) Spring stretched and block released

$$x(t) = \frac{v}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t})$$

(ii) The block given an initial positive velocity at equilibrium

(iii) Spring stretched out and the block given a velocity in the negative direction

$$x(t) = -\frac{v}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t}) + \frac{A}{\lambda_2 - \lambda_1} (\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t})$$

I would now like to tell you about the case when $\lambda_1 = \lambda_2$. This is known as the critically damped

case. Obviously this situation arises when $\frac{\gamma^2}{4} = \omega_0^2$. I can easily find solutions for such case if

I take the limit $\lambda_1 \rightarrow \lambda_2$ in the cases of heavy damping just studied. Please note that I cannot

straightaway take $\lambda_1 = \lambda_2$ in the expressions above because I am dividing by $(\lambda_2 - \lambda_1)$. Taking the limit gives for the three cases studied above

$$(i) \quad x(t) = A e^{-\lambda_1 t} (1 + \lambda_1 t)$$

$$(ii) \quad x(t) = v t e^{-\lambda_1 t}$$

$$(iii) \quad x(t) = (A - v t) e^{-\lambda_1 t}$$

As remarked above, the cases we have just discussed correspond to critical damping. In this

situation $\frac{\gamma^2}{4} = \omega_0^2$ and $\lambda_1 = \lambda_2 = -\frac{\gamma}{2}$. Mathematically, in this case there is only one solution ($e^{-\frac{\gamma}{2}t}$) that we get from the equation for λ because of its double root. The other solution is found to be $t e^{-\frac{\gamma}{2}t}$. That is precisely what we have found by taking appropriate limit.

Critically damped system and used when we want a system to return to its equilibrium position after receiving an impulse, although one is tempted to say that use a heavily damped system for this purpose. I would like you to understand this by carrying out the following exercise.

Exercise : The block on a damped spring-mass system is given an initial velocity v from equilibrium. Given a damping coefficient γ , plot the distance versus time graph for the critically and heavily damped cases. For ease of calculation take the heavy damping to be very large so

that $\frac{\gamma^2}{4} \gg \omega_0^2$ and make appropriate approximations.

Having discussed the heavily and critically damped systems, we move on to lightly

damped system. In such systems $\frac{\gamma^2}{4} < \omega_0^2$ so that

$$\lambda_1 = \frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2} = \frac{\gamma}{2} - i \omega_1$$

$$\lambda_2 = \frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2} = \frac{\gamma}{2} + i \omega_1$$

So the general solution is

$$x(t) = e^{-\frac{\gamma}{2}t} \left(A e^{i\omega_1 t} + B e^{-i\omega_1 t} \right)$$

Or equivalently

$$x(t) = e^{-\frac{\gamma}{2}t} (C \cos \omega_1 t + D \sin \omega_1 t)$$

In case when $\frac{\gamma^2}{4} \ll \omega_0^2$, it is called very light damping and in such case $\omega_1 \approx \omega_0$.

Let us now take a particular case when the block is stretched to distance A and is released from rest. I leave the details of the solution to be worked out by you. Here I give the final answer which is

$$x(t) = \frac{A\omega_0}{\omega_1} e^{-\frac{\gamma}{2}t} \cos(\omega_1 t - \phi) \quad \text{where} \quad \tan \phi = \frac{\gamma}{2\omega_1}$$

This solution is plotted schematically in figure 6. Notice how the maximum distance reached by the block decreases with time.

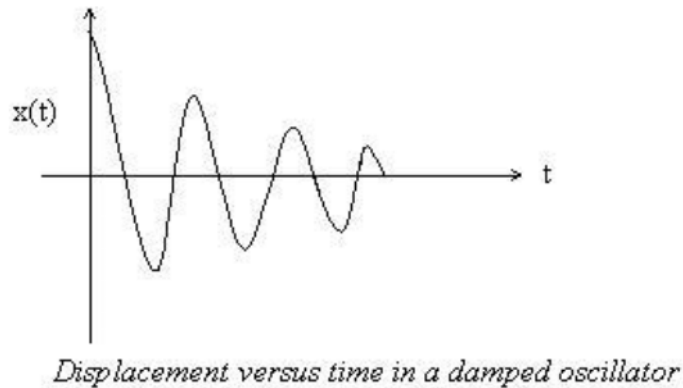


Figure 6

When we consider light damping, generally we are dealing with cases where we want the decay to be small. Thus within the time that the motion decays, there are many-many oscillations. Thus we can then write the displacement as

$$x(t) = \frac{A\omega_0}{\omega_1} e^{-\frac{\gamma}{2}t} \cos(\omega_1 t - \phi)$$

$$= A e^{-\frac{\gamma}{2}t} \cos \omega_0 t$$

because $\frac{\gamma^2}{4} \ll \omega_0^2$ implies that $\omega_1 \approx \omega_0$ and $\phi \approx 0$. The equation above is interpreted as the oscillation taking place with frequency ω_0 with time-dependent amplitude $A e^{-\frac{\gamma}{2}t}$.

Mathematically what this means is that $\frac{1}{\gamma} \gg \frac{1}{\omega_0}$ so there are two time scales in the problem. Let me now talk about the energy of the system. Since the amplitude is decreasing with time, the system is obviously losing energy. I want to calculate the rate of energy loss in the system. First,

there are many oscillations over the time interval of $\frac{1}{\gamma}$, which is also a very large time span. Further, the decay of the amplitude is very small over a few periods. This allows us to talk in terms of the average energy of the system. What it means is the energy averaged over a few cycles around a given instant. I now calculate it.

$$E(t) = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$$

Now use $x(t) = A e^{-\frac{\gamma}{2}t} \cos \omega_0 t$ to calculate this energy. It gives

$$E(t) = \frac{1}{2} m \left[\frac{\gamma^2}{4} A^2 \cos^2(\omega_0 t) + \omega_0^2 A^2 \sin^2(\omega_0 t) + \gamma \omega_0 A^2 \sin(\omega_0 t) \cos(\omega_0 t) \right] e^{-\gamma t}$$

$$+ \frac{1}{2} k A^2 \cos^2(\omega_0 t) e^{-\gamma t}$$

Now taking an average over a few cycles under the approximation that the exponentially decaying term be treated as roughly a constant over these cycles and neglecting the term proportional to γ^2 , I get

$$\langle E(t) \rangle = \frac{1}{2} k A^2 e^{-\gamma t}$$

where angular brackets denote the average energy. So the average energy decays exponentially for a lightly damped oscillator.

I now define the quality factor or Q for an oscillator. As mentioned earlier, we are interested in systems where $\gamma \ll \omega_0$; it is in such cases only that talking about Q makes sense. Q is defined as

$$Q = \frac{E_{\text{stored}}}{E_{\text{dissipated}} / \text{radian}}$$

$$= \frac{\omega_0}{\gamma}$$

High Q value for an oscillator means that there is very low leakage compared to the store energy.

Finally I summarize the lecture by telling you that we have covered the cases of heavy, critical and light damping in this lecture. You must have noticed that I have made a lot of graphs in this and the previous lecture. Please do that when you solve a problem. It will give you a feel for the system.

Harmonic oscillator III: Forced oscillations

In the previous two lectures, you have learnt about free harmonic oscillator and damped harmonic oscillator. In this lecture we study what happens when a harmonic oscillator is subjected to a force. The simplest case is when an oscillator is subjected to a constant force F . In that case nothing much takes place except that the equilibrium point gets shifted by (F/k) . You see an example of it when a mass is attached to a vertical spring. Mathematically we write

$$m\ddot{x} + kx = F$$

or $m\ddot{x} + kx - F = 0$

This can be written as

$$m\ddot{x} + k\left(x - \frac{F}{k}\right) = 0$$

for an undamped oscillator and

$$m\ddot{x} + \gamma\dot{x} + k\left(x - \frac{F}{k}\right) = 0$$

for a damped oscillator. Define a new variable $y = x - \frac{F}{k}$ so that the equation reads (I write only the undamped oscillator equation)

$$m\ddot{y} + ky = 0$$

This is the equation you are well familiar with. From its solution, that for x is written as

$$x = C \cos \omega_0 t + D \sin \omega_0 t + F/k$$

So the mass oscillates about $X = F/k$. I now take up an oscillator subjected to a time-dependent force.

A general time-dependent force $F(t)$ can always be decomposed into its Fourier components like

$$F(t) = \sum_n F_n \cos(n\omega t) \quad \text{so generally we study an oscillator subjected to a force of the form.}$$

$F(t) = F \cos \omega t$, where $\omega \neq \omega_0$ and F is the amplitude of the force. Let me start by first studying the motion of an undamped oscillator under such a force.

The equation of motion for an undamped oscillator under a time-periodic force is

$$m\ddot{x} + kx = F \cos(\omega t)$$

or equivalently

$$\ddot{x} + \omega_0^2 x = \frac{F}{m} \cos(\omega t)$$

The general solution is a combination of homogeneous part of the equation and a particular solution x_p . Thus

$$x(t) = C \cos \omega_0 t + D \sin \omega_0 t + x_p$$

Here you can check that

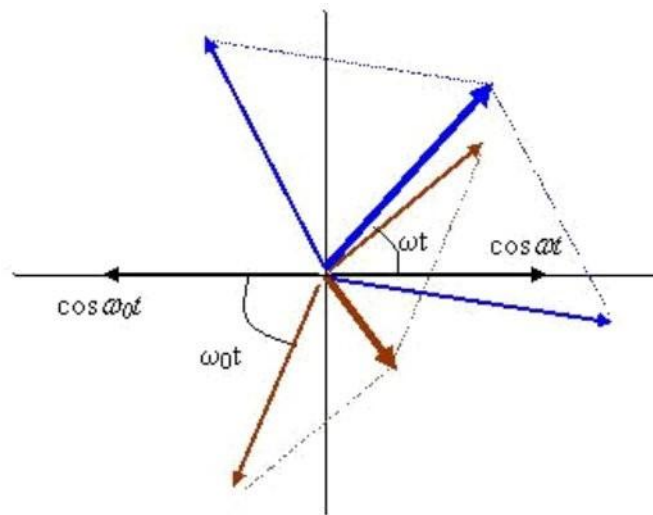
$$x_p = \frac{F/m}{(\omega_0^2 - \omega^2)} \cos \omega t$$

Let me start the oscillator from rest at equilibrium. It starts moving because of the applied force.

The initial conditions then are $x(0) = 0$ and $\dot{x}(0) = 0$. Under these conditions the solution comes out to be

$$x(t) = \frac{F}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t)$$

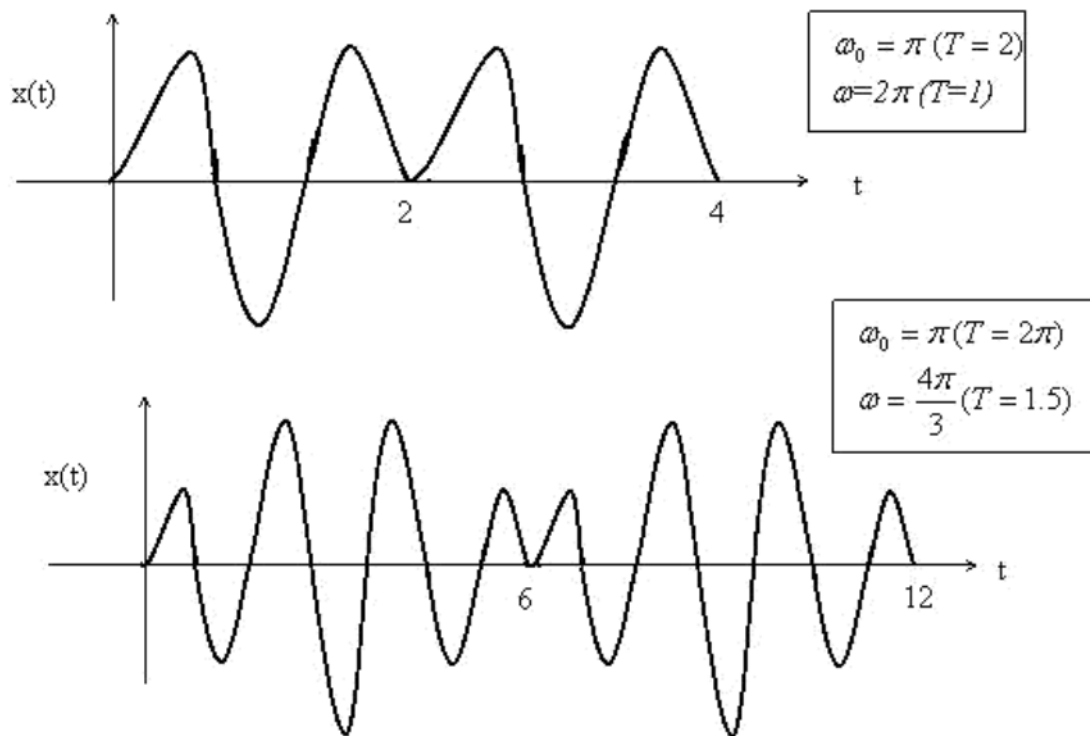
So the general solution is a combination of motion of two frequencies. The resulting motion can be represented on a phasor diagram by adding the two motions vectorially. This shown at $t = 0$ and two other different times in figure 1.



Motion of an undamped oscillator shown at $t=0$ (black) and two other different times, shown by brown and blue. The net displacement is shown by thick arrows.

Figure 1

As is clear from the figure, at $t = 0$, the net displacement is zero. As the time progresses, the displacement changes with the length of the rotating vector also changing with time. As an illustrative example, I take the frequency $\omega_0 = \pi (T_0 = 2)$, and two different frequencies, $\omega = 2\pi$ and $\omega = \frac{4\pi}{3}$ for the force. The resulting solutions are shown in figure 2.



*Forced oscillations of an undamped oscillator of frequency $\omega_0 = 2\pi$.
 Solutions are shown for two different frequencies of the applied force.*

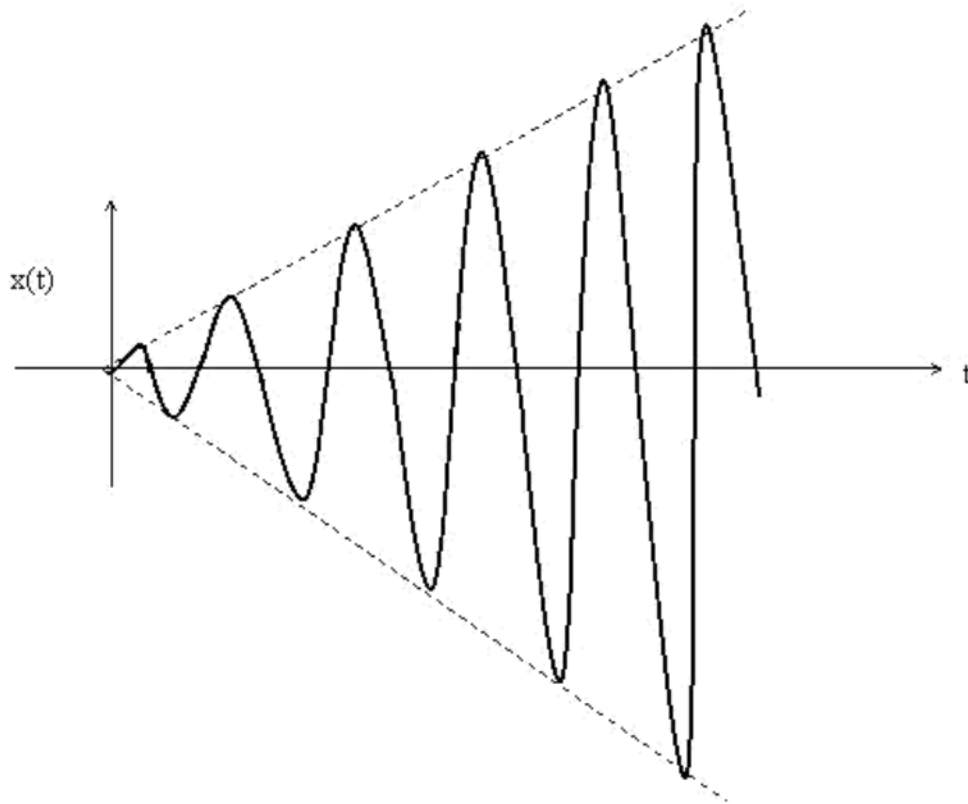
Figure 2

So you see from the figure above that the maximum displacement of oscillations keeps changing. This is what I had inferred from the phasor diagram also. The motion is still periodic and reminds us of the phenomena of beats.

Interesting is the case when $\omega = \omega_0$. However, I cannot put it directly in the formula because we are dividing by $(\omega_0^2 - \omega^2)$. So we have to take the limit $\omega \rightarrow \omega_0$. Let me substitute in the formula $\omega = (\omega_0 - \Delta)$ or $\omega = (\omega_0 + \Delta)$ and take $\Delta \rightarrow 0$. This leads to

$$x(t) = \frac{Ft}{2m\omega_0} \sin \omega_0 t$$

Thus the displacement keeps on increasing with time oscillating with the frequency of the oscillator. This is the phenomena of resonance. The corresponding plot of displacement is shown in figure 3.



Forced oscillations of an undamped at resonance

Figure 3

Having discussed forced oscillations for undamped oscillator, we now move on to study a damped oscillator moving under the influence of a periodic force. The equation of motion then is

$$m\ddot{x} + b\dot{x} + kx = F \cos(\omega t)$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F}{m} \cos(\omega t)$$

As earlier, the general solution of this equation is going to be the sum of the homogeneous and inhomogeneous part. So

$$x(t) = e^{-\frac{\gamma}{2}t} [C \cos \omega_1 t + D \sin \omega_1 t] + x_{\text{particular}}$$

As the time progresses $e^{-\frac{\gamma}{2}t}$ will make the homogeneous solution die down so finally the only solution remaining will be

$$x(t) = x_{\text{particular}}(t)$$

This is known as the steady state solution. Obviously it does not depend on the initial conditions. Let us now find this solution.

For the equation of motion

$$\ddot{x} + \gamma \dot{x} + \omega_o^2 x = \frac{F}{m} \cos(\omega t)$$

I assume a steady state solution of the form $A \cos \omega t$. But when substituted in the equation, this will give rise to a term containing $\sin \omega t$ because of $\gamma \dot{x}$ in the equation. So a general solution should be of the form.

$$x(t) = A \cos \omega t + B \sin \omega t$$

When substituted in the equation, this leads to

$$\begin{aligned} \cos \omega t [(\omega_o^2 - \omega^2)A + \gamma \omega B] + \sin \omega t [(\omega_o^2 - \omega^2)B - \gamma \omega A] &= \frac{F}{m} \cos \omega t \\ \Downarrow \\ (\omega_o^2 - \omega^2)B - \gamma \omega A &= 0 \quad \text{and} \quad (\omega_o^2 - \omega^2)A + \gamma \omega B = \frac{F}{m} \end{aligned}$$

These equations give

$$A = \frac{(\omega_o^2 - \omega^2) F / m}{(\omega_o^2 - \omega^2)^2 + \gamma^2 \omega^2} \quad \text{and} \quad B = \frac{\gamma \omega F / m}{(\omega_o^2 - \omega^2)^2 + \gamma^2 \omega^2}$$

So the general solution is

$$\begin{aligned}
x(t) &= \frac{(\omega_o^2 - \omega^2) F/m}{(\omega_o^2 - \omega^2)^2 + \gamma^2 \omega^2} \cos \omega t + \frac{\gamma \omega F/m}{(\omega_o^2 - \omega^2)^2 + \gamma^2 \omega^2} \sin \omega t \\
&= \frac{F/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + \gamma^2 \omega^2}} \left[\frac{(\omega_o^2 - \omega^2)}{\sqrt{(\omega_o^2 - \omega^2)^2 + \gamma^2 \omega^2}} \cos \omega t + \frac{\gamma \omega}{\sqrt{(\omega_o^2 - \omega^2)^2 + \gamma^2 \omega^2}} \sin \omega t \right] \\
&= \frac{F/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + \gamma^2 \omega^2}} \cos(\omega t - \phi)
\end{aligned}$$

where

$$\begin{aligned}
\sin \phi &= \frac{\gamma \omega}{\sqrt{(\omega_o^2 - \omega^2)^2 + \gamma^2 \omega^2}} & \cos \phi &= \frac{(\omega_o^2 - \omega^2)}{\sqrt{(\omega_o^2 - \omega^2)^2 + \gamma^2 \omega^2}} \\
\text{and} \quad \tan \phi &= \frac{\gamma \omega}{(\omega_o^2 - \omega^2)}
\end{aligned}$$

Thus after reaching steady state, the displacement lags behind the applied force by an angle ϕ

with $\tan \phi = \frac{\gamma \omega}{(\omega_o^2 - \omega^2)}$ and oscillates with an amplitude

$$A = \frac{F/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + \gamma^2 \omega^2}}$$

The oscillation frequency of steady-state solutions is obviously equal to the frequency of the applied force. A typical displacement and its shift with respect to the applied force are shown in figure 4.

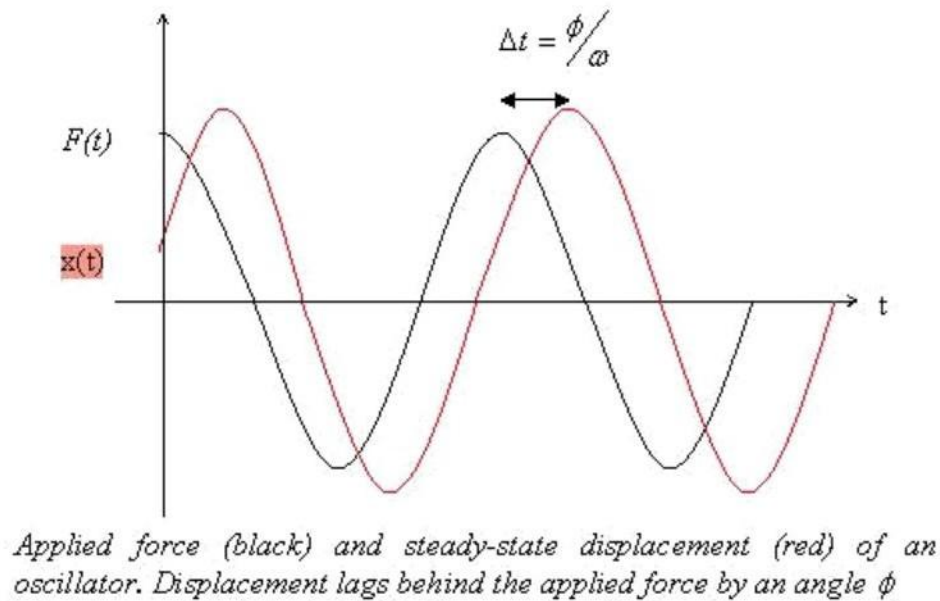


Figure 4

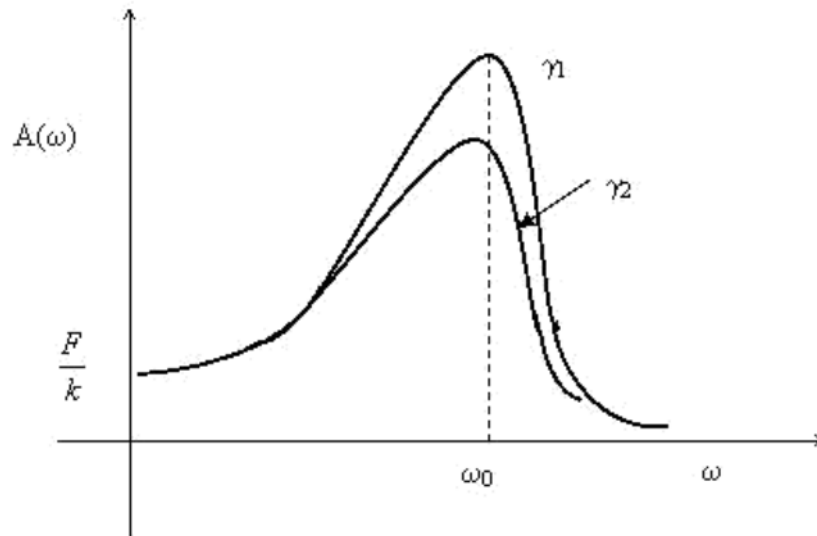
As far as getting the steady state solution for a forced damped oscillator is concerned, we are done. What we need to do now is to analyze the solution in different situations.

First of all we notice that irrespective of whether the system is lightly damped or heavily damped, it will always oscillate under an applied time-periodic force. Let us first consider the case of light damping and see how the amplitude varies with the applied frequency. The amplitude as a function of ω is given as

$$A(\omega) = \frac{F/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + \gamma^2 \omega^2}}$$

This amplitude goes to $\frac{F}{m\omega_o^2} = \frac{F}{k}$ as $\omega \rightarrow 0$. This is nothing but the stretch of the spring under

a constant force. For very large frequencies $A \approx \frac{F}{m\omega^2}$. In between the amplitude has a maximum at $\omega \approx \omega_o$ as is easily seen. So in this case, the amplitude as a function of frequency looks as shown in figure 5 for two different values of γ .

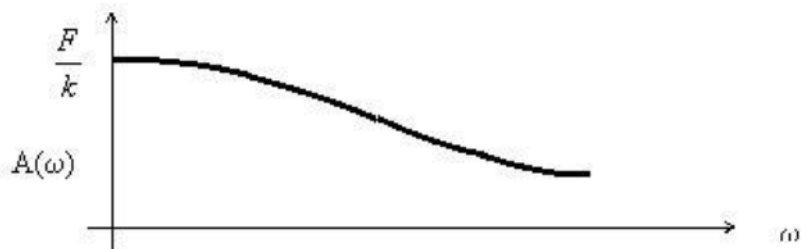


Amplitude as a function of frequency of applied force for two different values of γ with $\gamma_1 < \gamma_2$.

Figure 5

It is clear from the figure that the amplitude is maximum around $\omega = \omega_0$ which reminds us of the phenomenon of resonance for undamped oscillator. For large γ values the peak shifts to the left (lower frequency).

For heavy damping ($2\omega_0$) we do not see any amplitude maximum near $\omega = \omega_0$ but the system has large amplitude for low frequencies. A schematic plot of amplitude as a function of frequency looks like figure 6. It is evident that only for low frequencies the system oscillates with reasonable amplitude.



Amplitude as a function of frequency of applied force for a heavily damped oscillator.

Figure 6

What about the phase of the system with respect to the applied force? I leave this as an exercise for you to plot the phase of displacement as a function of frequency.

Next I discuss how much power is absorbed by the system to maintain its oscillations.

Power absorption in a forced damped oscillator : Since a damped system has a retardation force opposing its motion, it dissipates energy. For it to maintain a steady-state the applied force constantly supplies energy to it. It is this power that I now calculate. Power given to the system is

$\vec{F} \cdot \vec{v} = F \dot{x}$ since I am considering a one dimensional system. Otherwise I would have taken the dot product between the force and the velocity. The calculation proceeds as follows

$$\begin{aligned} P &= F \dot{x} \\ &= -F \cos \omega t \times \frac{F/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + \gamma^2 \omega^2}} \omega \sin(\omega t - \phi) \\ &= -\frac{F^2 \omega}{m \sqrt{(\omega_o^2 - \omega^2)^2 + \gamma^2 \omega^2}} [\cos \omega t \sin \omega t \cos \phi - \cos^2 \omega t \sin \phi] \end{aligned}$$

Since the average of $\cos^2 \theta$ or $\sin^2 \theta$ over a cycle is $1/2$ and that of $\cos \theta \sin \theta$ zero, the average the last expression with respect to time over one cycle gives

$$P = \frac{F^2 \omega}{2m \sqrt{(\omega_o^2 - \omega^2)^2 + \gamma^2 \omega^2}} \sin \phi$$

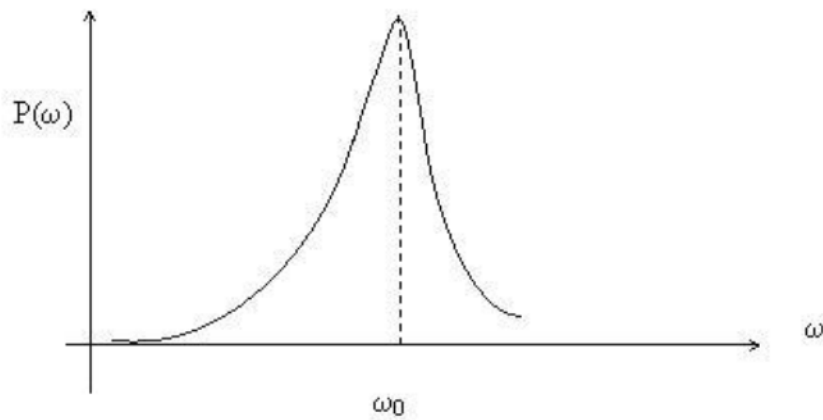
This is the average power being supplied to the system to maintain its steady-state. The same can also be obtained by realizing that in steady-state the power given to the system is the same as power dissipated by it. Power dissipated is the drag force ($F_{drag} = -bv = -\gamma m v$) times the velocity. This is therefore calculated as follows:

$$\begin{aligned} P &= -bv^2 \\ &= -\gamma m \cdot \frac{F^2/m^2}{[(\omega_o^2 - \omega^2)^2 + \gamma^2 \omega^2]} \omega^2 \sin^2(\omega t - \phi) \end{aligned}$$

Taking its time average over a cycle then gives the average dissipated power

$$P = \frac{-F^2 \gamma \omega^2}{2m[(\omega_o^2 - \omega^2)^2 + \gamma^2 \omega^2]}$$

which is the same result as obtained above. The negative sign shows that this is the energy lost, and produced heat due to the friction in the system. Since the amplitude of the motion is largest when the force has a frequency close to the natural frequency of a system, it is expected that the power loss will also be maximum near that frequency. I have plotted the power dissipated in a forced damped harmonic oscillator in figure 7.



Power dissipation in a forced damped oscillator as a function of frequency of the applied force.

Figure 7

The curve peaks at ω_0 so the power absorption is indeed maximum at the resonance frequency.

Finally I relate the Q factor of a damped oscillator with the power versus frequency curve given above. To do this let us see at what frequency does the power absorption is $\frac{1}{2}$ of its peak value. The calculation, in which we make the frequency-dependent factor in the expression for power dimensionless and equate it to $\frac{1}{2}$, is given below

$$\begin{aligned} \frac{\gamma^2 \omega^2}{[(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2]} &= \frac{1}{2} \\ 2\gamma^2 \omega^2 &= (\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2 \\ \text{or } \omega_0^2 - \omega^2 &= \pm \gamma \omega \end{aligned}$$

Solving this equation for the frequency ω under the approximation of light-damping gives

$$\omega = \omega_0 + \frac{\gamma}{2} \quad \text{and} \quad \omega = \omega_0 - \frac{\gamma}{2}$$

The frequency width from $\omega_o - \frac{\gamma}{2}$ to $\omega_o + \frac{\gamma}{2}$ is known as full width at half maximum (FWHM) and its value is γ . Thus the quality factor can also be interpreted as

$$Q = \frac{\omega_o}{\gamma} = \frac{\text{resonance frequency}}{\text{full width half maximum}}$$

This pretty much sums up what I want to tell you about forced oscillations. I want to point out that we have focused here strictly on the steady-state solutions for the damped oscillator. However, before steady-state is reached, the system goes through transient motion, which is also important to understand in designing of systems.